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Inference Mechanisms, Systems of Fuzzy Relational Equations and the Additive Interpretations of Rule Bases

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Abstract

Fuzzy inference systems are studied from the point of view of systems of fuzzy relation equations. A fundamental interpolation condition is considered to be a crucial point of study in choosing proper inference method as well as a proper interpretation of a fuzzy rule base. The paper aims at additive interpretations and investigates their utilization from a theoretical point of view while their approximation abilities are already justified.

1 INTRODUCTION

Fuzzy inference systems are very frequently used in many areas of applications e.g. decision making, control or pattern recognition. A typical feature of the fuzzy inference systems is the fact that they deal with imprecisely given information. In fact, there are two main parts of the fuzzy inference systems - an imprecise (say expert) knowledge about a modelled system which is coded in the form of a *fuzzy rule base* and a rule determining the way how to deduce conclusions from imprecise observations which is called *inference mechanism*.

A fuzzy rule base is given by a set of n fuzzy rules

$$\textbf{IF } x \text{ is } \mathcal{A}_i \textbf{ THEN } y \text{ is } \mathcal{B}_i \quad i = 1, \ldots, n \tag{1}$$

where the linguistic expressions $\mathcal{A}_i, \mathcal{B}_i$ are represented by fuzzy sets $\mathbf{A}_i \underset{\sim}{\subset} X, \mathbf{B}_i \underset{\sim}{\subset} Y$, respectively.

An inference mechanism, roughly said, is understood as a mapping which assigns to a fuzzy set $\mathbf{A} \underset{\sim}{\subset} X$ a conclusion $\mathbf{B} \underset{\sim}{\subset} Y$ defined as an image of $\mathbf{A}$ under a fuzzy relation $\mathbf{R} \underset{\sim}{\subset} X \times Y$ interpreting fuzzy rule base (1). Mostly, the image is defined by the sup-$*$ composition

$$\mathbf{B} = \mathbf{A} \circ_* \mathbf{R} \tag{2}$$

which comes from the compositional rule of inference (CRI) i.e.

$$(\mathbf{A} \circ_* \mathbf{R})(y) = \bigvee_{x \in X} (\mathbf{A}(x) * \mathbf{R}(x, y)) \tag{3}$$

where $*$ is an arbitrary left continuous t-norm.

Another possibility, how to define the image under a fuzzy relation is the inf$\rightarrow_*$ composition

$$\mathbf{B} = \mathbf{A} \looparrowright_* \mathbf{R} \tag{4}$$

given as follows

$$(\mathbf{A} \looparrowright_* \mathbf{R})(y) = \bigwedge_{x \in X} (\mathbf{A}(x) \rightarrow_* \mathbf{R}(x, y)) \tag{5}$$

where $\rightarrow_*$ is a residuation operation derived from a t-norm $*$.

Fuzzy rule base (1) defines a partial mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ and therefore either we deal with composition (2) or composition (4), the fundamental interpolation condition claiming that from $\mathbf{A}_i$ we deduce exactly $\mathbf{B}_i$ should be kept.

This task leads to the problematic of systems of fuzzy relational equations which aims especially at investing solvability conditions of the systems of fuzzy relation equations, see [4]. However, besides solutions given by

$$\hat{\mathbf{R}}_*(x, y) = \bigwedge_{i=1}^{n} (\mathbf{A}(x) \rightarrow_* \mathbf{B}(y)),$$

$$\check{\mathbf{R}}_*(x, y) = \bigvee_{i=1}^{n} (\mathbf{A}(x) * \mathbf{B}(y)),$$

searching for appropriate solutions $\mathbf{R}$ has not been focused so much yet [9]. Moreover, there are many fuzzy inference techniques which were proposed either heuristically or because of good approximation abilities but still not well studied from the fuzzy interpolation point of view.

This contribution focuses on systems using the so called *additive interpretations* [12] of fuzzy rule bases which are motivated by Takagi-Sugeno models [13], singleton models, fuzzy transform [10] or some neuro-fuzzy systems. A formalization of such additive interpretations can be seen in the additive normal form introduced in [11] and then studied in [1, 2].

The main goal of the paper is to search for a solution of (2) or (4) in the following form

$$\mathbf{R}_*^{\oplus}(x,y) = \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) * \mathbf{B}_i(y)) \tag{6}$$

where $\oplus$ is the Łukasiewicz t-conorm. It means that we search for specific conditions related to operations in inference mechanisms and fuzzy sets in a rule base which ensure keeping the interpolation condition and so a proper use of the additive interpretations.

2 PRELIMINARIES IN FRE's

This section focuses on systems of fuzzy relation equations and recalls basic results especially from [3, 5, 7]. Interpolation condition leads to the following systems of fuzzy relation equations

$$\mathbf{A_i} \circ_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n \tag{7}$$

and

$$\mathbf{A_i} \looparrowright_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n. \tag{8}$$

Now, we recall necessary and sufficient conditions of the solvability of the previous systems of fuzzy relation equations.

Theorem 1 *The system*

$$\mathbf{A_i} \circ_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n \tag{9}$$

is solvable if and only if $\hat{\mathbf{R}}_*$ *is its solution and then* $\hat{\mathbf{R}}_*$ *is its greatest solution.*

Theorem 2 *The system*

$$\mathbf{A_i} \looparrowright_* \mathbf{R} = \mathbf{B}_i \quad i = 1, \ldots, n \tag{10}$$

is solvable if and only if $\check{\mathbf{R}}_*$ *is its solution and then* $\check{\mathbf{R}}_*$ *is its least solution.*

The following theorem collects solvability conditions of system (7) from [5] and of system (8) published in [8].

Theorem 3 *Let* $\mathbf{A}_i$*,* $i = 1, \ldots, n$*, be normal. Then* $\check{\mathbf{R}}_*$ *is a solution to (7) and* $\hat{\mathbf{R}}_*$ *is a solution to (8) if and only if the following semi-partition condition*

$$\bigvee_{x \in X} (\mathbf{A}_i(x) * \mathbf{A}_j(x)) \leq \bigwedge_{y \in Y} (\mathbf{B}_i(y) \leftrightarrow_* \mathbf{B}_j(y)), \quad i, j = 1, \ldots, n \tag{11}$$

holds.

3 ADDITIVE SOLUTIONS TO

Let us recall a definition of the generalized orthogonality [11] which will be very useful in the latter.

Definition 1 We say that $\mathbf{A}_i \underset{\sim}{\subset} X,\ i = 1, \ldots, n$ keep the orthogonality condition if

$$\bigoplus_{\substack{i=1 \\ i \neq j}}^{n} \mathbf{A}_i(x) = 1 - \mathbf{A}_j(x). \tag{12}$$

At first, we consider an additive solution to the inf$\rightarrow_*$ composition.

Theorem 4 *Let (8) be solvable, $\mathbf{A}_i$, $i = 1, \dots, n$ be normal and keep the orthogonality condition. Then $\mathbf{R}_*^{\oplus}$ is a solution to (8).*

PROOF: Let $j \in \{1, \dots, n\}$ be an arbitrary fixed subindex and let

$$\mathbf{B}(y) = \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

then

$$\mathbf{B}(y) \geq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigvee_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y)))$$

and if (8) is solvable then $\check{\mathbf{R}}_*$ is its solution i.e. $\mathbf{B} \subseteq \mathbf{B}_j$. On the other hand, because of the antitonicity of residuation operations and because of the orthogonality (see [11]) the following

$$\bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigoplus_{i=1}^{n} (\mathbf{A}_i(x) * \mathbf{B}_i(y))) \leq \bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y)))$$

where $\otimes$ is the Łukasiewicz t-norm, holds. Let $x' \in X$ such that $\mathbf{A}_j(x') = 1$ then

$$\bigwedge_{x \in X} (\mathbf{A}_j(x) \rightarrow_* \bigwedge_{i=1}^{n} (\mathbf{A}_i(x) \rightarrow_{\otimes} \mathbf{B}_i(y))) \leq \mathbf{A}_j(x') \rightarrow_* (\mathbf{A}_j(x') \rightarrow_{\otimes} \mathbf{B}_j(y))$$

which yields $\mathbf{B} \supseteq \mathbf{B}_j$.

□

Corollary 1 *Let $\mathbf{A}_i$, $i = 1, \dots, n$ be normal and semi-partition condition (11) holds. If $\mathbf{A}_i$ keep the orthogonality condition then $\mathbf{R}_*^{\oplus}$ is a solution to (8).*

Now, we recall the definition of a stronger t-norm, see [6].

Definition 2 Let $*$ and $\blacktriangle$ be t-norms. Then we say that $\blacktriangle$ is a stronger t-norm than $*$ ($* \leq \blacktriangle$) if $\forall x_1, x_2 \in [0, 1]: \; x_1 * x_2 \leq x_1 \blacktriangle x_2$.

Theorem 5 *Let (8) be solvable, $\mathbf{A}_i$, $i = 1, \dots, n$ be normal and keep the orthogonality condition. Then $\mathbf{R}_{\blacktriangle}^{\oplus}$ where $* \leq \blacktriangle$ is a solution to (8).*

PROOF: The proof is analogous to the proof of Theorem 4 and therefore it is omitted. □

Corollary 2 *Let $\mathbf{A}_i$, $i = 1, \dots, n$ be normal and semi-partition condition (11) holds. If $\mathbf{A}_i$ keep the orthogonality condition then $\mathbf{R}_{\blacktriangle}^{\oplus}$ where $* \leq \blacktriangle$ is a solution to (8).*

A possibility of taking an additive interpretation of a fuzzy rule base together with the inf$\rightarrow_*$ composition as an inference method was partially clarified by the previous theorems and corollaries. In the latter, we investigate a possibility to connect an additive interpretation of a fuzzy rule base to the sup-$*$ composition, which is a bit more complicated. As a tool of verifying this potentiality we again use the systems of fuzzy relation equation, particularly system (7).

Theorem 6 *Let $\mathbf{A}_i$, $i = 1, \dots, n$ be normal, $\mathbf{A}_i$ keep the orthogonality condition and let $* \leq \otimes$ where $\otimes$ is the Łukasiewicz t-norm. If the semi-partition condition holds then $\mathbf{R}_*^{\oplus}$ is a solution to (7).*

PROOF: By direct assignments one gets $\mathbf{B} \supseteq \mathbf{B}_j$. On the other hand

$$\bigvee_{x\in X}(\mathbf{A}_j(x) * \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) * \mathbf{B}_i(y))) \le \bigvee_{x\in X}(\mathbf{A}_j(x) * \bigwedge_{i=1}^{n}(\mathbf{A}_i(x) \to_{\otimes} \mathbf{B}_i(y))) \le$$

$$\le \bigvee_{x\in X}(\mathbf{A}_j(x) * (\mathbf{A}_j(x) \to_{\otimes} \mathbf{B}_j(y))) \le \bigvee_{x\in X}(\mathbf{A}_j(x) \otimes (\mathbf{A}_j(x) \to_{\otimes} \mathbf{B}_j(y)))$$

and because $a * (a \to b) \le b$ for any $*$ and its residuation operations $\to$ we immediately get $\mathbf{B} \subseteq \mathbf{B}_j$. □

Theorem 6 requires to use a t-norm which is even weaker than the Łukasiewicz one while it is already a very weak t-norm, see [6]. So, for practical applications, perhaps only the case when $* = \otimes$ worths mentioning. In this case, the Łukasiewicz t-norm is used for both, the sup-$\otimes$ composition as an inference method and for the interpretation of a fuzzy rule base by the fuzzy relation $\mathbf{R}^{\oplus}_{\otimes}$. Of course, the t-norm $\otimes$ has to appear in the semi-partition as well.

The result is strengthened by the following one.

Theorem 7 *Let $\mathbf{A}_i$, $i = 1, \ldots, n$ be normal, $\mathbf{A}_i$ keep the orthogonality condition and let $* \le \otimes$ as well as $* \le \blacktriangle$. If the following semi-partition condition*

$$\bigvee_{x\in X}(\mathbf{A}_i(x) \blacktriangle \mathbf{A}_j(x)) \le \bigwedge_{y\in Y}(\mathbf{B}_i(y) \leftrightarrow_{\blacktriangle} \mathbf{B}_j(y)) \tag{13}$$

holds then $\mathbf{R}^{\oplus}_{\blacktriangle}$ is a solution to (7).

PROOF: Based on the fact that for $* \le \blacktriangle$ semi-partition condition (13) w.r.t. $\blacktriangle$ implies semi-partition condition (11) w.r.t. $*$ we observe that $\check{\mathbf{R}}_*$ is a solution to (7). And since $\mathbf{R}^{\oplus}_{\blacktriangle} \supseteq \check{\mathbf{R}}_*$ we conclude $\mathbf{B} \supseteq \mathbf{B}_j$.

The other side is proved analogously to the proof of Theorem 6. □

Theorem 7 allows us to deal with a t-norm weaker or equal to the Łukasiewicz one only in the sup-$*$ composition as an inference method but the interpretation of a fuzzy rule base can be build w.r.t. a t-norm $\blacktriangle$ which is stronger. The cost we have to pay for it is that the semi-partition condition has to be kept also w.r.t. the t-norm $\blacktriangle$ what is a stronger assumption of the semi-partition condition for the stronger t-norm.

To demonstrate an influence to real applications of the results introduced above, let us consider a simple example. Let $\mathbf{A}_i$, $i = 1, \ldots, n$ keep the Ruspini condition

$$\sum_{i=1}^{n}\mathbf{A}_i(x) = 1, \quad \forall x \in X \tag{14}$$

which is very often required for antecedent fuzzy sets when constructing a fuzzy rule based systems. Then fuzzy sets obviously keep the orthogonality condition as well as the following semi-partition condition

$$\bigvee_{x\in X}(\mathbf{A}_i(x) \otimes \mathbf{A}_j(x)) \le \bigwedge_{y\in Y}(\mathbf{B}_i(y) \leftrightarrow_{\otimes} \mathbf{B}_j(y)) \tag{15}$$

since on the left hand side there is always 0 for distinct fuzzy sets $\mathbf{A}_i$ and $\mathbf{A}_j$. Then because of Corollary 2 it is enough to add the normality of fuzzy sets $\mathbf{A}_i$ and to take a stronger t-norm than the Łukasiewicz one e.g. product to be sure that the following interpretation of a fuzzy rule base

$$\mathbf{R}^{\oplus}_{\odot} = \bigoplus_{i=1}^{n}(\mathbf{A}_i(x) \odot \mathbf{B}_i(y)) = \sum_{i=1}^{n}\mathbf{A}_i(x)\mathbf{B}_i(y) \tag{16}$$

will keep the interpolation condition while we use inf-$\to_{\otimes}$ composition as an inference method. Moreover, if the fuzzy sets $\mathbf{B}_i$ are such that the following semi-partition

$$\bigvee_{x\in X}(\mathbf{A}_i(x) \odot \mathbf{A}_j(x)) \le \bigwedge_{y\in Y}(\mathbf{B}_i(y) \leftrightarrow_{\odot} \mathbf{B}_j(y)) \tag{17}$$

holds, then due to Theorem 7 the interpolation condition will be kept even with the sup-$\otimes$ composition as an inference method.

4 CONCLUSION

The paper dealt with the so called additive interpretations of fuzzy rule bases which is a formalization of various fuzzy approximation techniques. A theoretical investigation of the additive interpretation connected to two most often used fuzzy inference method was introduced and conditions yielding a proper use of the additive interpretations were obtained. As a tool for a verification of the proper use, systems of fuzzy relation equations were studied. This lead to a retrieval of new solutions of systems of fuzzy relation equation while up to now, usually only two main solutions $\hat{\mathbf{R}}$ and $\check{\mathbf{R}}$ were studied.

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References

[1] Daňková, M. - Štěpnička, M.: Fuzzy transform as an additive normal form. Fuzzy Sets and Systems, 157, 2006, 1024-1035.

[2] Daňková, M. - Štěpnička, M.: Genetic algorithms in fuzzy approximation. In: Proc. of the Joint 4th EUSFLAT2005 and 11th LFA2005, 2005, 651-656.

[3] DeBaets, B.: Analytical solution methods for fuzzy relation equations. In: D. Dubois, H. Prade (Eds.) The Handbook of Fuzzy Set Series, Vol. 1, Academic Kluwer Publ., Boston, 2000, 291-340.

[4] Gottwald, S.: Generalized Solvability behaviour for systems of fuzzy equations. In: V. Novák and I. Perfilieva, (Eds.) Discovering the World with Fuzzy Logic (Studies in Fuzzines and Soft Computing series). Springer-Verlag, Heidelberg, New York, pp. 401-430, 2000.

[5] Klawon, F.: Fuzzy points, fuzzy relations and fuzzy functions. In: V. Novák, I. Perfilieva (Eds.) Discovering the World with Fuzzy Logic. Advances in Soft Computing, Springer Physica-Verlag, Heidelberg, 2000, 431-453.

[6] Klement, E.P. - Mesiar, R. - Pap, E.: Triangular Norms. Kluwer, Boston, Dordrecht, 2000.

[7] Klir, G.J. - Yaun, B.: Fuzzy Sets and Fuzzy Logic. Prentice Hall, New Jersey, 1995.

[8] Nosková, L.: Systems of fuzzy relation equation with inf$\rightarrow$ composition: solvability and solutions, Journal of Electrical Engineering, 12(s), 2005, 69-72.

[9] Perfilieva, I.: Fuzzy function as an approximate solution to a system of fuzzy relation equations, Fuzzy Sets and Systems, 147, 2004, 363383.

[10] Perfilieva, I.: Fuzzy transforms: Theory and applications, Fuzzy Sets and Systems, 157, 2006, 993-1023.

[11] Perfilieva, I.: Normal forms in BL and ŁΠ algebras of functions, Soft Computing, 8, 2004, 291-298.

[12] Štěpnička M. - Valášek, R.: Generating a fuzzy rule base with an additive interpretation. In: Proc. of 16th IFAC World Congress'05 in Prague - Volume C2 - Cognition and Control (AI, Fuzzy, Neuro, Evolut.), Elsevier, Prague, 2006, 233-238.

[13] Takagi, T. - Sugeno, M.: Fuzzy identification of systems and its applications to modeling and control, IEEE Transactions on Systems, Man and Cybernetics, 15, 1985, 116-132.