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System of fuzzy relation equations with inf-$\rightarrow$ composition in semi-linear spaces: maximal solutions

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Abstract

The problem of solvability of a system of equations with inf-$\rightarrow$ composition has been considered in semilinear spaces. We used the fact that a set of solutions is determined by maximal solutions and therefore, we focused on characterization of them. At first, full sets of maximal solutions of single equations in the case of a finite universe and different assumptions on an underlying algebra have been described. Depending on the ordering of a support set, either necessary or sufficient conditions, or criteria of being a maximal solution have been obtained. Then maximal solutions of a system are build from maximal solutions of single equations. Many examples of single equations and systems are considered and full sets of solutions are found for them.

Keywords: semi-linear spaces, system of linear-like equations, fuzzy relation equations, maximal solution

1 Introduction

Linear models of processes, systems or events are attractive for researchers because of their transparent structure. In this respect, a demonstration that a certain process is linear or, at least, semi-linear always meets with success. Besides classical examples of actually linear systems there are other examples, e.g., discrete event systems [3], which are usually analyzed using semilinear structures (in this case, with the help of the max-plus algebra [4, 20]).

Linear behavior of fuzzy systems has been recently discovered in [8]. Two cases may occur: a behavior of a system is characterized by fuzzy IF-THEN rules (expert knowledge and similar) or it is characterized by a set of input-output pairs of fuzzy sets (monitoring, collecting knowledge, etc.). In the second case, a problem of solving a respective system of fuzzy relation equations arises [9, 15, 16, 21]. However, it has not been realized yet (besides, probably in [5]) that the mentioned problem is similar to the problem of solving systems of linear equations. In this paper, we will show that systems of fuzzy relation equations can be considered as systems of linear-like equations in a semilinear space over BL-algebra. We will introduce two types of systems of fuzzy relation equations (dependently on the used operations). One is the well known sup-$*$ the second one, which was not sufficiently studied in the literature is inf-$\rightarrow$. We will focus on the latter and characterize complete sets of solutions which constitute a $\wedge$-semi-lattice.

2 Preliminaries

2.1 BL-algebra

The concept of BL-algebra has been introduced in [13] as the algebra of operations which correspond to connectives of basic fuzzy logic (BL). In the same sense as BL generalizes boolean logic we can say that BL-algebra generalizes boolean algebra. This appears in the extension of the set of boolean operations by two operations which constitute the, so called, adjoined couple. The following definition summarizes definitions originally introduced in [13].

Definition 1
A BL-algebra *is an algebra*

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, 0, 1)$ *is a lattice with* 0 *and* 1,

(ii) $(L, *, 1)$ *is a commutative semigroup with unit* 1,

(iii) $*$ *and* $\rightarrow$ *(implication) establish an adjoint pair, i.e.*
$z \leq (x \rightarrow y)$ *iff* $x * z \leq y$ *for all* $x, y, z \in L$,

(iv) for all $x, y \in L$
$x *(x \rightarrow y) = x \wedge y$,
$(x \rightarrow y) \vee (y \rightarrow x) = 1$.

Other operation of $\mathcal{L}$ is the unary $\neg$ (negation) which is defined by

$$\neg x = x \rightarrow 0.$$

Besides boolean algebra, the following algebras are other examples of BL-algebra.

Example 1 (Gödel algebra)

$$\mathcal{L}_G = \langle [0, 1], \vee, \wedge, \rightarrow_G, 0, 1 \rangle$$

where the multiplication $* = \wedge$ *and*

$$x \rightarrow_G y = \begin{cases} 1 & \textit{if} \quad x \leq y, \\ y & \textit{if} \quad y < x, \end{cases} \qquad \neg_G x = \begin{cases} 1 & \textit{if} \quad x = 0, \\ 0 & \textit{if} \quad x > 0. \end{cases}$$

Example 2 (Product (Goguen) algebra)

$$\mathcal{L}_P = \langle [0, 1], \vee, \wedge, \odot, \rightarrow_P, 0, 1 \rangle$$

where the multiplication $\odot = \cdot$ *is the ordinary product of reals and*

$$x \rightarrow_P y = \begin{cases} 1 & \textit{if} \quad x \leq y, \\ \frac{y}{x} & \textit{if} \quad y < x, \end{cases} \qquad \neg_P x = \begin{cases} 1 & \textit{if} \quad x = 0, \\ 0 & \textit{if} \quad x > 0. \end{cases}$$

The following two laws specify Product algebras among BL ones:

$$\textit{if } z \neq 0 \textit{ then } x * z = y * z \Rightarrow x = y \qquad \text{(cancelation)}$$
$$x \wedge \neg x = 0 \qquad \text{(contradiction)}$$

Example 3 (Łukasiewicz algebra)

$$\mathcal{L}_L = \langle [0,1], \vee, \wedge, \otimes, \rightarrow_L, 0, 1 \rangle$$

where

$$x \otimes y = 0 \vee (x + y - 1), \qquad x \rightarrow_L y = 1 \wedge (1 - x + y), \qquad \neg_L x = 1 - x.$$

In the sequel, we will often use the following property of BL-algebra:

$$x \leq y \quad \text{iff} \quad (x \rightarrow y) = 1.$$

2.2 Semirings and semilinear spaces

In this subsection we will recall the definition of a semiring, then give a definition of a semilinear space and consider some examples of both structures. Although a notion of a semiring is known almost 100 years (originating in Dedekind's studies), a comprehensive study has been published only recently [11]. The definition below is taken from [11].

Definition 2
A semiring $\mathcal{R} = \langle R, +, \cdot, 0, 1 \rangle$ is an algebra where

(SR1) $\langle R, +, 0 \rangle$ is a commutative monoid,

(SR2) $\langle R, \cdot, 1 \rangle$ is a monoid,

(SR3) for all $a, b, c \in R$

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a,$$

(SR4) for all $a \in R$

$$0 \cdot a = a \cdot 0 = 0.$$

A semiring is commutative if $(R, \cdot, 1)$ is a commutative monoid.

A typical example of a commutative semiring is a set $\mathbb{N}$ of non-negative integers with addition and multiplication.

Remark 1
In our construction the last property (SR4) is not required. Thus further on, a semiring will be considered without (SR4). However, whenever (SR4) holds true, we will put a notice saying that the semiring has an annihilating zero.

Below we will give examples of semirings which can be taken as reducts of BL-algebras.

Example 4
1. Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1 \rangle$ be a BL-algebra. Then its $\vee$-reduct

$$\mathcal{L}_\vee = \langle L, \vee, *, 0, 1 \rangle$$

is a commutative semiring with annihilator.

2. Let again $\mathcal{L}$ be a BL-algebra. The following algebra ($\wedge$-reduct of $\mathcal{L}$) is a commutative semiring:

$$\mathcal{L}_\wedge = \langle L, \wedge, *, 1, 1\rangle.$$

Let us remark that the semirings in Example 4 are the idempotent semirings because their operation of "addition" is idempotent (see e.g. also [4, 14]).

The following definition is again taken from [11].

Definition 3
Let $A \neq \emptyset$ be a set of elements and $\mathcal{R} = \langle R, +, \cdot, 0, 1\rangle$ a semiring. We say that A is a (left) semimodule over $\mathcal{R}$ if two operations are defined:

(a) addition $+$ such that for each two elements $a, b \in A$ there is a uniquely determined element $a + b \in A$ called their sum,

(b) multiplication $\cdot$ by an element from R such that for any $a \in A$ and $p \in R$ there is a uniquely determined element $p \cdot a$ called their product.

These operations fulfil the following properties for all $a, b, c \in A$ and $p, q \in R$:

(SL1) $a + b = b + a$,

(SL2) $a + (b + c) = (a + b) + c$,

(SL3) there exists the (neutral) element $\mathbf{0} \in A$ such that $a + \mathbf{0} = a$,

(SL4) $p \cdot (a + b) = p \cdot a + p \cdot b$,

(SL5) $(p + q) \cdot a = p \cdot a + q \cdot a$,

(SL6) $p \cdot (q \cdot a) = (p \cdot q) \cdot a$,

(SL7) $1 \cdot a = a$.

A right semimodule over $\mathcal{R}$ may be defined analogously.

The notion of a semilinear space first appeared in [18] in connection with power algebras over semirings. It has been used later in [8] to explain fuzzy systems and their principles.

Definition 4
Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a BL-algebra and $\mathcal{R}$ its semiring reduct. Then an $\mathcal{R}$-semimodule over $\mathcal{L}$ is called a semilinear space.

The elements of a semilinear space are called vectors and elements of L scalars.

Example 5
1. Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a BL-algebra on L, $\mathcal{L}_\vee = \langle L, \vee, *, 0, 1\rangle$ its semiring reduct. Let us consider the set of all n-dimensional vectors $A = L^n$, $n \geq 1$, and define

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 \vee b_1, \ldots a_n \vee b_n),$$
$$p \cdot (a_1, \ldots, a_n) = (p * a_1, \ldots, p * a_n)$$

where $p \in L$. The neutral element in A is the vector $(0, \ldots, 0)$. It is easy to see that L^n is a semilinear space over $\mathcal{L}_\vee$.

2. *Let $X \neq \emptyset$, $\mathcal{L}$ be a BL-algebra on L and $\mathcal{L}_\vee = \langle L, \vee, *, 0, 1\rangle$ its semiring reduct. Let us consider the set of all L-valued functions $A = L^X$ and put*

$$A(x) + B(x) = A(x) \vee B(x),$$
$$p \cdot A(x) = p * A(x)$$

where $p \in L$. The neutral element in A is the function which is identically equal to 0. It is easy to see that L^X is a semilinear space over $\mathcal{L}_\vee$.

3. *Let $\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ be a BL-algebra on L, $\mathcal{L}_\wedge = \langle L, \wedge, *, 1, 1\rangle$ its semiring reduct. Let us take the set of all n-dimensional vectors $A = L^n$, $n \geq 1$, and define*

$$(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 \wedge b_1, \ldots a_n \wedge b_n),$$
$$p \cdot (a_1, \ldots, a_n) = (p * a_1, \ldots, p * a_n)$$

where $p \in L$. The neutral element in A is the vector $(1, \ldots, 1)$. It is easy to see that L^n is a semilinear space over $\mathcal{L}_\wedge$.

Remark 2
The semilinear space in our first example has been first studied in [6] under the name fuzzy subspace.

2.3 Fuzzy sets, fuzzy relations and their compositions

More than forty years has passed since Lotfi A. Zadeh [22] published the first paper about fuzzy sets [22]. Now the theory of fuzzy sets demonstrates its maturity and penetrates almost all mathematical subjects. According to the original conception, a fuzzy set is identified with its characteristic function which maps a universe into the unit interval. Goguen [10] extended the original construction by using an arbitrary bounded distributive lattice instead of the unit interval.

Let $\mathcal{L}$ be a BL-algebra with a support L. We consider a characteristic function of a fuzzy set as a mapping from a universe, say U into L and identify a fuzzy set with its membership function. Further on, we will consider fuzzy sets on a finite universe $U = \{u_1, \ldots, u_m\}$, $m \geq 1$, which comprise a semilinear space L^m over $\mathcal{L}_\vee$, see Example 5.

If $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$, $m, n \geq 1$, are finite universes then a fuzzy subset of Cartesian product $U \times V$ is called a (binary) *fuzzy relation*. Any fuzzy relation $R \in L^{U \times V}$ determines a homomorphism h_R from semilinear space L^U to semilinear space L^V via the so called sup-$*$ composition:

$$h_R(A)(v_j) = (R \circ A)(v_j) = \bigvee_{i=1}^{m} (R(u_i, v_j) * A(u_i)),\ j = 1, \ldots, n.$$

Mapping $h_R : L^U \longrightarrow L^V$ has the *residual* ([2, 5]) $g_R : L^V \longrightarrow L^U$ being determined via the inf-$\rightarrow$ composition ([1]):

$$g_R(B)(u_i) = (R \triangleright B)(u_i) = \bigwedge_{j=1}^{n} (R(u_i, v_j) \rightarrow B(v_j)),\ i = 1, \ldots, m.$$

Two systems of fuzzy relation equations are connected with h_R or g_R:

$$R \circ X = B \qquad (\text{ or } R \triangleright X = B)$$

where R, B are known parameters and X is unknown. In the literature on fuzzy sets and systems they are referred to as *a system of fuzzy relation equations with* sup-$*$ *composition* and *a system of fuzzy relation equations with* inf-$\longrightarrow$ *composition*. However, the first system is better investigated in what concerns solvability, characterization of solutions, even characterization approximate solutions (see e.g. [7, 9, 12, 17, 19, 21]). On the other hand, not much attention has been paid to the problem of a solvability of a system of fuzzy relation equations with inf-$\longrightarrow$ composition. Generally, it has been discovered that each systems is solvable if and only if a certain relation (different for different cases) gives a solution. Moreover, a set of solutions is a semi-lattice with the largest (the case of sup-$*$ composition) or the least (the case of inf-$\longrightarrow$ composition) element ([9, 21]).

The aim of this contribution is to characterize a full set of solutions to the system with inf-$\longrightarrow$ composition in the case of a finite universe and different assumptions on a BL-algebra. Moreover, instead of dealing with fuzzy sets and fuzzy relations we reformulated the problem as a problem of solvability of a residual system of equations in a semilinear space. On this theoretical platform we will characterize a set of solutions by characterizing a set of maximal solutions.

2.4 Systems of Linear-like Equations in Semilinear Space of Vectors

In what follows, we fix a complete BL-algebra with a support L and consider L^m and L^n, $m, n \geq 1$, as semilinear spaces over $\mathcal{L}_\vee$, see Example 5. Let $A = (a_{ij})$ be a $(n \times m)-$matrix and $\mathbf{b} = (b_1, \dots, b_n)$ vector, both have components from L. The following system of equations is considered with respect to an unknown vector $\mathbf{x} = (x_1, \dots, x_m)$:

$$\bigwedge_{j=1}^{m} (a_{ij} \rightarrow x_j) = b_i, \quad i = 1, \dots n, \tag{1}$$

or in a vector form

$$A \triangleright \mathbf{x} = \mathbf{b}.$$

It has been mentioned in Preliminaries that systems of equations similar to (1) have been considered in the literature devoted to fuzzy sets and systems, see e.g. [1, 7, 9, 15]. From these sources we took the following result.

Proposition 1
System (1) is solvable if and only if the vector $\check{\mathbf{x}} = (\check{x}_j)$ *is its least solution where*

$$\check{x}_j = \bigvee_{i=1}^{n} (a_{ij} * b_i), \quad j = 1, \dots, m.$$

Moreover, if $\mathbf{x}^1$ and $\mathbf{x}^2$ are solutions to (1) then $\mathbf{x}^1 \wedge \mathbf{x}^2$ where the operation $\wedge$ is taken componentwise is a solution too. Therefore, the set of solutions of (1) form a $\wedge$-semilattice with "zero" element. Thus, to obtain a set of solutions, it would be sufficient to obtain all maximal solutions. We will continue as follows: in Section 3 we will consider a system consisting of one equation for which a complete characterization of a set of solutions is given. Then in Section 4 we will investigate a general system of equations and characterize a set of solutions as well.

3 The Case of a Single Relation Equation

In a semilinear space L^m we consider the following equation with respect to an unknown vector $\mathbf{x} = (x_1, \ldots, x_m) \in L^m$:

$$\bigwedge_{j=1}^{m} (a_j \to x_j) = b. \tag{2}$$

Vector $\mathbf{a} = (a_1, \ldots, a_m) \in L^m$ and $b \in L$ are supposed to be known.

The following fact resulting from Proposition 1 will be used in the sequel: equation (2) is solvable if and only if the vector $\check{\mathbf{x}} = (\check{x}_j)$ is its least solution where

$$\check{x}_j = a_j * b, \quad j = 1, \ldots, m.$$

Moreover, if $\mathbf{x}^1$ and $\mathbf{x}^1$ are solutions to (2) then $\mathbf{x}^1 \wedge \mathbf{x}^2$ where the operation $\wedge$ is taken componentwise is a solution too. Therefore, the set of solutions of (2) forms a $\wedge$-semilattice with "zero" element. Thus, aiming to get a set of solutions, it would be sufficient to get all maximal solutions. We will proceed as follows:

- investigate and exclude conditions on given data $(a_1, \ldots, a_m)$, b which do not lead to solvability of (2);
- explicitly characterize conditions formulated in terms of given data $(a_1, \ldots, a_m)$, b which guarantee existence of maximal solutions and by this, solvability of (2);
- characterize all maximal solutions.

3.1 Necessary and Sufficient Conditions of Solvability of a Single Relation Equation

We may easily obtain the following necessary condition of solvability of (2).

Lemma 1
Let equation (2) be solvable, then the following condition

$$b \geq \bigwedge_{j=1}^{m} \neg a_j \tag{3}$$

necessarily holds.

PROOF: By the assumption, (2) is solvable, and let $\mathbf{x} = (x_j)$ be its solution. Then we obtain:

$$b = \bigwedge_{j=1}^{m} (a_j \to x_j) \geq \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j.$$

□

As we see, solvability of (2) leads to a certain relation between b and $\bigwedge_{j=1}^{m} \neg a_j$. We prove even more.

Theorem 1

(i) Equation (2) is unsolvable if $b < \bigwedge_{j=1}^{m} \neg a_j$;

(ii) Equation (2) is solvable if $b = \bigwedge_{j=1}^{m} \neg a_j$.

PROOF:

(i) Assume that $b < \bigwedge_{j=1}^{m} \neg a_j$, but equation (2) is solvable. Then $\check{\mathbf{x}} = (\check{x}_j)$ is its least solution, i.e. $\check{x}_j = a_j * b \leq a_j * \neg a_j = 0$ or $\check{x}_j = 0$, j=1,..., m. In this case,

$$b = \bigwedge_{j=1}^{m} (a_j \to \check{x}_j) = \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j > b$$

which is a contradiction.

(ii) Suppose that $b = \bigwedge_{j=1}^{m} \neg a_j$ holds true. Then we easily verify that $\check{x}_k = 0$, $k = 1, \ldots, n$. Indeed, $\check{x}_k = a_k * b = a_k * \bigwedge_{j=1}^{m} \neg a_j \leq a_k * \neg a_k = 0$. It turned out that $\check{\mathbf{x}} = (0, \ldots, 0)$ is a solution to (2). Indeed,

$$\bigwedge_{j=1}^{m} (a_j \to \check{x}_j) = \bigwedge_{j=1}^{m} (a_j \to 0) = \bigwedge_{j=1}^{m} \neg a_j = b.$$

Therefore, (2) is solvable. □

From Lemma 1 and Theorem 1 we cannot formulate a criterion of solvability. This is because we cannot prove in general that, if $b \geq \bigwedge_{j=1}^{m} \neg a_j$ then (2) is solvable. The assertion in this form holds true under special conditions on a lattice ordering given in Proposition 2.

3.2 The Set of Solutions of a Single Relation Equation

By Proposition 1, equation (2) is solvable if and only if the vector

$$\check{\mathbf{x}} = (\check{x}_j), \quad \check{x}_j = a_j * b, \quad j = 1, \ldots, m$$

is its solution and moreover, $\check{\mathbf{x}}$ is the least solution. Therefore, in order to obtain the whole set of solutions of (2) it is sufficient to characterize all maximal solutions.

We assume that the necessary condition of solvability is fulfilled, i.e. $b \geq \bigwedge_{j=1}^{m} \neg a_j$ and distinguish the following three cases depending on the value of b.

3.2.1 The set of solutions in the case $b = 1$

It is easy to see that in this case equation (2) is always solvable and the greatest solution is given by

$$\mathbf{x} = (1, \ldots, 1).$$

Together with the fact that $\check{\mathbf{x}} = (a_1, \ldots, a_m)$ is the least solution, we conclude that all vectors $\mathbf{x} = (x_1, \ldots, x_m) \in L^m$ fulfilling $a_j \leq x_j$ are solutions.

3.2.2 The set of solutions in the case $b < 1$ and $b = \neg a_k$

Let $b < 1$ and for some $k = 1, \ldots, m$, $b = \neg a_k$. Under the given assumptions, equation (2) is solvable by Theorem 1. The following lemmas and theorems characterize maximal solutions of (2).

Theorem 2
Suppose that $b < 1$ and $b = \neg a_k$ for some $k = 1, \ldots, m$. Then the following vector

$$\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{c_k}_{k}, \ldots, 1)$$

where $c_k = \sup\{s : a_k \wedge s = 0\}$ gives the respective maximal solution to (2).

PROOF: First, we verify that $\hat{\mathbf{x}}^k$ is a solution to (2). This follows from

(a) $a_k \wedge c_k = a_k \wedge \bigvee_{a_k \wedge s = 0} s = \bigvee_{a_k \wedge s = 0} (a_k \wedge s) = 0;$

(b) $a_k \wedge c_k = 0 \Leftrightarrow a_k * (a_k \to c_k) = 0 \Leftrightarrow a_k \to c_k \leq a_k \to 0;$

(c) $a_k \to c_k \geq a_k \to 0;$

(d) $\bigwedge_{j=1}^{m} (a_j \to \hat{x}_j^k) = a_k \to c_k = \neg a_k = b.$

As a side result we have proved that in a BL-algebra (not necessarily complete) the following two equalities are logically equivalent:

$$a \to b = a \to 0 \quad \text{iff} \quad a \wedge b = 0. \tag{4}$$

Second, we prove that $\hat{\mathbf{x}}^k$ is a maximal solution. Therefore, we assume that the vector $\mathbf{x}$ solves (2) and is greater than $\hat{\mathbf{x}}^k$ so that

$$\mathbf{x} = (1, \ldots, \underbrace{x_k}_{k}, \ldots, 1) \text{ and } x_k > c_k.$$

Obviously, $a_k \to x_k \geq a_k \to c_k$ or $a_k \to x_k \geq a_k \to 0$. Let us prove that the equality $a_k \to x_k = a_k \to 0$ is not possible. Indeed, this equality implies $a_k * (a_k \to x_k) = 0$, or $a_k \wedge x_k = 0$, or $x_k \leq c_k$ which is a contradiction with the second assumption on $\mathbf{x}$.

Therefore,

$$\bigwedge_{j=1}^{m} (a_j \to x_j) = a_k \to x_k > a_k \to 0 = b$$

which contradicts the first assumption on $\mathbf{x}$ and so, the second claim is proved. □

Corollary 1
Suppose that the assumption of Theorem 2 holds true and the underlying BL-algebra $\mathcal{L}$ is linearly ordered. Then the following vector

$$\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{0}_{k}, \ldots, 1)$$

gives the respective maximal solution to (2).

PROOF: From the assumption that $b < 1$ and for some $k = 1, \ldots, m$, $b = \neg a_k$, it follows that $a_k \neq 0$. Therefore, in a linearly ordered BL-algebra $c_k = \sup\{s : a_k \wedge s = 0\} = 0$.

Remark 3
Although we generally assume that a BL-algebra is complete, the above given Corollary can be proved without this assumption.

□

The theorem below provides with a criterion of constituting a maximal solution with the additional assumption $b = \bigwedge_{j=1}^{m} \neg a_j$.

Theorem 3
Let $\mathcal{L}$ be linearly ordered and let us have in equation (2) the following: $b < 1$, $b = \bigwedge_{j=1}^{m} \neg a_j$ and $b = \neg a_k$ for $k = k_1, \ldots, k_l$, $1 \leq l \leq m$. Then $\hat{\mathbf{x}}$ is a maximal solution if and only if

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}^k = (1, \ldots, \underbrace{0}_{k}, \ldots, 1)$$

for some $k \in \{k_1, \ldots, k_l\}$.

PROOF: *Necessity.* We will prove that each maximal solution belongs to the set $\{\hat{\mathbf{x}}^k \mid k \in \{k_1, \ldots, k_l\}$ where $\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{0}_{k}, \ldots, 1)$. Suppose that $\mathbf{x} = (x_1, \ldots, x_m)$ is an arbitrary solution to (2). Then $\bigwedge_{j=1}^{m}(a_j \to x_j) = b$.

By $b = \bigwedge_{j=1}^{m} \neg a_j = \neg a_k$ for all $k = k_1, \ldots, k_l$, we have: $\neg a_j > b$ for $j \notin \{k_1, \ldots, k_l\}$ and therefore, $a_j \to x_j \geq a_j \to 0 > b$. Then $\bigwedge_{j=1}^{m}(a_j \to x_j) = a_{\tilde{k}} \to x_{\tilde{k}}$ for some $\tilde{k} \in \{k_1, \ldots, k_l\}$.

Thus for this $\tilde{k}$, $a_{\tilde{k}} \to x_{\tilde{k}} = b$ and $b = \neg a_{\tilde{k}} = a_{\tilde{k}} \to 0$. By (4), $a_{\tilde{k}} \wedge x_{\tilde{k}} = 0$. Since $b < 1$ and $\mathcal{L}$ is linearly ordered, this implies $x_{\tilde{k}} = 0$. Therefore, $\mathbf{x} \leq \hat{\mathbf{x}}^{\tilde{k}}$ and $\hat{\mathbf{x}}^{\tilde{k}}$ is a maximal solution to (2).

Sufficiency. Under the given assumptions and by Corollary 1, vector $(1, \ldots, \underbrace{0}_{k}, \ldots, 1$ is a maximal solution to (2) for each $k = k_1, \ldots, k_l$. □

The corollary below characterizes the whole set of solutions of (2).

Corollary 2
Suppose that the assumptions of Theorem 3 hold true and $\mathbf{x} = (x_j)$ is a solution to (2). Then there exists one $k \in \{k_1, \ldots, k_l\}$, such that for any $j = 1, \ldots, m$, it is true that

(i) if $j = k$ then $x_j = 0$,

(ii) if $j \neq k$ then $x_j \geq a_j * b$.

PROOF: The least solution to (2) is $\check{\mathbf{x}} = (\check{x}_j)$ with $\check{x}_j = a_j * b, \quad j = 1, \ldots, m$. For $j = k_1, \ldots, k_l$ we easily have:

$$\check{x}_j = a_j * b = a_j * \neg a_j = 0.$$

Then the statement follows from Theorem 3. □

Remark 4
It is easy to see that if the assumptions of Theorem 3 hold true for exactly one k then the set of solutions of (2) is linearly ordered and $\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{0}_{k}, \ldots, 1)$ is the largest solution.

Example 6
(a) *Let a semilinear space $[0,1]^n$, $n = 3$, be considered over Łukasiewicz algebra (see Example 3) on $[0,1]$. The following equation*

$$(1.0 \to_L x_j) \wedge (0.1 \to_L x_j) \wedge (0.3 \to_L x_j) = 1.0$$

is solvable according to Subsection 3.2.1. The least solution $\check{\mathbf{x}} = (1, 0.1, 0.3)$, the largest solution $\hat{\mathbf{x}} = (1.0, 1.0, 1.0)$ and therefore, each vector $\mathbf{x} = \{x_1, x_2, x_3\} \in [0,1]^3$ with components: $x_1 = 1, x_2 \geq 0.1, x_3 \geq 0.3$ gives a solution.

(b) *In the same semilinear space $[0,1]^n$, $n = 3$, over Łukasiewicz algebra, the following equation*

$$(0.3 \to_L x_j) \wedge (0.5 \to_L x_j) \wedge (0.2 \to_L x_j) = 0.5$$

is considered. In this case, conditions of Theorem 3 and Remark 4 are fulfilled. Therefore, the set of solutions is linearly ordered, the least solution $\check{\mathbf{x}} = (0,0,0)$, the largest solution $\hat{\mathbf{x}} = (1,0,1)$ and each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0,1]^3$ satisfies: $x_1 \geq 0, x_2 = 0, x_3 \geq 0$.

3.2.3 The set of solutions in the case $b < 1$ and $b > \bigwedge_{j=1}^m \neg a_j$

Let (2) be solvable and additionally: $b < 1$ and $b > \bigwedge_{j=1}^m \neg a_j$. We will characterize maximal solutions of (2) in this case and under two additional assumptions on a BL-algebra $\mathcal{L}$:

(i) $\mathcal{L}$ is linearly ordered;

(ii) the conditional cancelation law holds: $0 < a * x \leq a * y \Rightarrow x \leq y$,

Typical examples are the Product algebra $\mathcal{L}_P$ (see Example 2) and Łukasiewicz algebra $\mathcal{L}_Ł$ (see Example 3).

Theorem 4
Let $\mathcal{L}$ fulfil the assumptions of Subsection 3.2.3 and $b < 1$ be such that for some $k = 1, \ldots, m$, $b > \neg a_k$. Then the following vector

$$\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{a_k * b}_{k}, \ldots, 1)$$

gives a maximal solution to (2).

PROOF: Let the condition of the theorem hold for k, $1 \leq k \leq m$. Then obviously, $a_k > 0$ and $a_k * b > 0$. Let us verify that $\hat{\mathbf{x}}^k$ is a solution to (2). Indeed by the conditional cancelation law,

$$\bigwedge_{j=1}^{m} (a_j \to \hat{x}_j^k) = a_k \to a_k * b = \sup\{c \mid a_k * c \leq a_k * b\} = \sup\{c \mid c \leq b\} = b.$$

We will prove that this solution is maximal. On the contrary, we assume that a vector $\mathbf{x} = (x_j)$ solves (2) and is greater than $\hat{\mathbf{x}}^k$ so that

$$\mathbf{x} = (1, \ldots, \underbrace{x_k}_{k}, \ldots, 1) \text{ and } x_k > a_k * b$$

and

$$\bigwedge_{j=1}^{m} (a_j \to x_j) = a_k \to x_k = b.$$

Therefore,

$$x_k > a_k * (a_k \to x_k) = a_k \wedge x_k.$$

By linear ordering of L, we conclude that $a_k \wedge x_k = a_k$ so that $x_k > a_k$ and $a_k \to x_k = 1$. This contradicts the equality $a_k \to x_k = b$ and by this, proves the maximality of $\hat{\mathbf{x}}^k$. □

Similarly to proofs of Theorem 3 and Corollary 2 we obtain a criterion of constituting a maximal solution and a characterization of all solutions to (2).

Theorem 5
Let $\mathcal{L}$ fulfil the assumptions of Subsection 3.2.3.

(a) *Suppose that in equation (2) we have: $b < 1$, $b > \bigwedge_{j=1}^{m} \neg a_j$ and $b \geq \neg a_k$ for $k = k_1, \ldots, k_l$, $1 \leq l \leq m$. Then $\hat{\mathbf{x}}$ is a maximal solution if and only if*

$$\hat{\mathbf{x}}^k = (1, \ldots, \underbrace{a_k * b}_{k}, \ldots, 1).$$

for some $k \in \{k_1, \ldots, k_l\}$.

(b) *If moreover $\mathbf{x} = (x_j)$ is an arbitrary solution to (2) then there exists one $k \in \{k_1, \ldots, k_l\}$, such that*

(i) $x_k = a_k * b$,

(ii) $x_j \geq a_j * b$, $j \neq k$.

PROOF: We will prove the claim (a) and (b) follows immediately.

Necessity. We show that each maximal solution belongs to the set $\{\hat{\mathbf{x}}^k \mid k \in \{k_1, \dots, k_l\}$ where $\hat{\mathbf{x}}^k = (1, \dots, \underbrace{a_k * b}_{k}, \dots, 1)$. Suppose that $\mathbf{x} = (x_1, \dots, x_m)$ is an arbitrary solution to (2). Then $\bigwedge_{j=1}^m (a_j \to x_j) = b$. By $b > \bigwedge_{j=1}^m \neg a_j$ and $b \geq \neg a_k$ for $k = k_1, \dots, k_l$, we have: $\neg a_j > b$ and therefore, $a_j \to x_j \geq a_j \to 0 > b$ for $j \notin \{k_1, \dots, k_l\}$. Then $\bigwedge_{j=1}^m (a_j \to x_j) = a_{\tilde{k}} \to x_{\tilde{k}}$ for some $\tilde{k} \in \{k_1, \dots, k_l\}$.

Thus for this $\tilde{k}$, $a_{\tilde{k}} \to x_{\tilde{k}} = b$. Now two cases are possible:

(i) $b = \neg a_{\tilde{k}}$ where in the same way as it has been proved in Theorem 3, we conclude that $x_{\tilde{k}} = 0 = a_{\tilde{k}} * b$;

(ii) $b > \neg a_{\tilde{k}}$ where by the conditional cancelation and other assumptions on $\mathcal{L}$, $x_{\tilde{k}} = a_{\tilde{k}} * b$.

In both cases $\mathbf{x} \leq \hat{\mathbf{x}}^{\tilde{k}}$ and $\hat{\mathbf{x}}^{\tilde{k}}$ is a maximal solution to (2).

Sufficiency. Under the given assumptions and by Corollary 1 and Theorem 4, vector $(1, \dots, \underbrace{a_k * b}_{k}, \dots, 1)$ is a maximal solution to (2) for each $k = k_1, \dots, k_l$.

□

Summarizing results which have been proved in Section 3 (Lemma 1, Theorems 2, 3, 4, 5 and their corollaries), we come to the following criterion.

Proposition 2

Assume that $\mathcal{L}$ is linearly ordered and the cancelation law holds. Then equation (2) is solvable if and only if $b \geq \bigwedge_{j=1}^m \neg a_j$. Moreover, each solution of (2) is less or equal of a respective maximal solution and

(i) if $b = 1$ then $\mathbf{x} = (1, \dots, 1)$ is the only maximal solution,

(ii) if $b < 1$ and $b = \bigwedge_{j=1}^m \neg a_j$ then the number of maximal solutions is finite and all of them are described in Theorem 3,

(iii) if $b < 1$ and $b > \bigwedge_{j=1}^m \neg a_j$ then the number of maximal solutions is finite and all of them are described in Theorem 5.

Example 7

- *In the semilinear space $[0,1]^n$, $n = 3$, over Product algebra (see Example 2), the following equation*

$$(0.3 \to_P x_j) \wedge (0.5 \to_P x_j) \wedge (0.8 \to_P x_j) = 0.5$$

is considered. In this case, conditions of Theorem 5 are fulfilled. Therefore, there are three maximal solutions to this equation: $(0.15, 1, 1)$, $(1, 0.25, 1)$ *and* $(1, 1, 0.4)$. *The least solution is* $\check{\mathbf{x}} = (0.15, 0.25, 0.4)$.

Each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0,1]^3$ *satisfies: either* $\check{\mathbf{x}} \leq \mathbf{x} \leq (0.15, 1, 1)$ *or* $\check{\mathbf{x}} \leq \mathbf{x} \leq (1, 0.25, 1)$ *or* $\check{\mathbf{x}} \leq \mathbf{x} \leq (1, 1, 0.4)$.

- *The same equation as above is considered in the semilinear space $[0,1]^n$, $n = 3$, over Łukasiewicz algebra $\mathcal{L}_L$ (see Example 3). Again conditions of Theorem 5 are fulfilled, but (differently to the above given example) there are two maximal solutions to this equation: $(1,0,1)$ and $(1,1,0.3)$. The least solution is $\check{\mathbf{x}} = (0,0,0.3)$.*

  Each solution $\mathbf{x} = \{x_1, x_2, x_3\} \in [0,1]^3$ satisfies: either $\check{\mathbf{x}} \le \mathbf{x} \le (1,0,1)$ or $\check{\mathbf{x}} \le \mathbf{x} \le (1,1,0.3)$.

4 System of Relation Equations: General Information

In this section, a general system of equations will be investigated and a full set of its solutions will be characterized. We recall the system (1)

$$\bigwedge_{j=1}^{m}(a_{ij} \to x_j) = b_i, \quad i = 1, \dots n$$

for which a $(n \times m)-$ matrix $A = (a_{ij}) \in L^{n\times m}$ and a vector $\mathbf{b} = (b_1, \dots, b_n) \in L^n$ are given. We will be looking for an unknown vector $\mathbf{x} = (x_1, \dots, x_m) \in L^m$. The following fact from Proposition 1 will be used in the sequel: equation (2) is solvable if and only if

$$\check{\mathbf{x}} = (\check{x}_j), \quad \check{x}_j = \bigvee_{i=1}^{n}(a_{ij} * b_i), \quad j = 1, \dots, m$$

is its least solution.

If the system (1) is solvable then each of its equations is solvable as well. Therefore, the lemma analogous to Lemma 1 establishes the necessary condition of solvability.

Lemma 2
Let the system (1) be solvable, then for each $i = 1, \dots n$, the following condition

$$b_i \ge \bigwedge_{j=1}^{m} \neg a_{ij} \tag{5}$$

necessarily holds.

Lemma 3
The system (1) is unsolvable if $b_i < \bigwedge_{j=1}^{m} \neg a_{ij}$ for some $i,\ i = 1, \dots, n$.

PROOF: The proof easily follows from Theorem 1. □

Proposition 3
The system (1) is solvable if for all $i = 1, \dots, n$, $b_i = \bigwedge_{j=1}^{m} \neg a_{ij}$.

PROOF: Suppose that for all $i = 1, \dots, n$, $b_i = \bigwedge_{j=1}^m \neg a_{ij}$ holds true. Then we easily verify that $\check{x}_k = 0$, $k = 1, \dots, n$. Indeed,

$$\check{x}_k = \bigvee_{i=1}^n (a_{ik} * b_i) = \bigvee_{i=1}^n \left(a_{ik} * \bigwedge_{j=1}^m \neg a_{ij} \right) \leq \bigvee_{i=1}^n (a_{ik} * \neg a_{ik}) = 0.$$

Moreover, $\check{\mathbf{x}} = (0, \dots, 0)$ is a solution to (1). Indeed,

$$\bigwedge_{j=1}^m (a_{ij} \to \check{x}_j) = \bigwedge_{j=1}^m (a_{ij} \to 0) = \bigwedge_{j=1}^m \neg a_{ij} = b_i.$$

Therefore, (1) is solvable. □

4.1 Maximal solutions of a System of Relation Equations

With respect to the fact that $\check{\mathbf{x}}$ is the least solution of (1), it is sufficient to characterize all maximal solutions (if any) in order to obtain the whole set of solutions. The following theorem gives a general method of the construction of a maximal solution to (1).

Theorem 6
Let system (1) be solvable and $\mathbf{x}^i = (x_{i1}, \dots, x_{im}) \in L^m$ be a maximal solution to the i-th equation in (1). Let moreover,

$$\bigwedge_{i=1}^n \mathbf{x}^i \geq \check{\mathbf{x}} \tag{6}$$

hold true. Then

$$\bar{\mathbf{x}} = \bigwedge_{i=1}^n \mathbf{x}^i \tag{7}$$

is a maximal solution to (1).

PROOF: Suppose that the system (1) is solvable and $\mathbf{x}^i = (x_{i1}, \dots, x_{im}) \in L^m$ is a maximal solution to the i-th equation in (1). Then each component of $\bar{\mathbf{x}}$ is equal to

$$\bar{x}_j = \bigwedge_{i=1}^n x_{ij}, \quad j = 1, \dots, m.$$

First, we verify that $\bar{\mathbf{x}}$ is a solution to (1). Actually, we verify that $\bar{\mathbf{x}}$ is a solution to each i-th equation in (1), $i = 1, \dots, n$. Indeed,

$$\begin{aligned} \bigwedge_{j=1}^m (a_{ij} \to \bar{x}_j) &= \bigwedge_{j=1}^m \left(a_{ij} \to \bigwedge_{l=1}^n x_{lj} \right) = \bigwedge_{l=1}^n \bigwedge_{j=1}^m (a_{ij} \to x_{lj}) = \\ &= b_i \wedge \bigwedge_{l=1, l \neq i}^n \bigwedge_{j=1}^m (a_{ij} \to x_{lj}) \leq b_i. \end{aligned}$$

On the other side by (6),

$$\bigwedge_{j=1}^{m}(a_{ij} \rightarrow \bar{x}_j) \geq \bigwedge_{j=1}^{m}(a_{ij} \rightarrow \check{x}_j) = b_i.$$

Both inequalities confirm that

$$\bigwedge_{j=1}^{m}(a_{ij} \rightarrow \bar{x}_j) = b_i$$

so that $\bar{\mathbf{x}}$ solves each i-th equation in (1).

In order to verify that $\bar{\mathbf{x}}$ is a maximal solution, let us take an arbitrary solution $\mathbf{x}$ and notice that for each $i = 1, \ldots, n$, $\mathbf{x} \leq \mathbf{x}^i$. The latter is due to the fact that $\mathbf{x}$ solves each i-th equation in (1) as well. Therefore,

$$\mathbf{x} \leq \bigwedge_{i=1}^{n} \mathbf{x}^i$$

which confirms our claim. □

Example 8

Let a semilinear space $[0,1]^3$ be considered over Łukasiewicz algebra and the system (1) be given by the matrix

$$A = \begin{pmatrix} 1.0 & 0.0 & 0.4 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

and the vector

$$\mathbf{b} = \begin{pmatrix} 1.0 \\ 0.5 \end{pmatrix}.$$

(a) The first equation

$$(1.0 \rightarrow_L x_j) \wedge (0.0 \rightarrow_L x_j) \wedge (0.4 \rightarrow_L x_j) = 1.0$$

is analogous to that which has been considered in Example 6, case (a). The least solution is $\check{\mathbf{x}}^1 = (1, 0, 0.4)$ and the largest solution is $\hat{\mathbf{x}}^1 = (1, 1, 1)$.

(b) The second equation

$$(0.3 \rightarrow_L x_j) \wedge (0.5 \rightarrow_L x_j) \wedge (0.2 \rightarrow_L x_j) = 0.5$$

has been considered in Example 6, case (b). The least solution $\check{\mathbf{x}}^2 = (0, 0, 0)$, the largest solution $\hat{\mathbf{x}}^2 = (1, 0, 1)$.

(c) Vector $\check{\mathbf{x}} = (1, 0, 0.4)$, componentwise obtained by $\check{\mathbf{x}} = \check{\mathbf{x}}^1 \vee \check{\mathbf{x}}^2$, gives the least solution to the system.

(d) Vector $\hat{\mathbf{x}} = (1, 0, 1)$, componentwise obtained by $\hat{\mathbf{x}} = \hat{\mathbf{x}}^1 \wedge \hat{\mathbf{x}}^2$, is greater than $\check{\mathbf{x}} = (1, 0, 0.4)$ and by Theorem 6, gives the maximal solution to the considered system.

Conclusion. *Each solution* $\mathbf{x} = \{x_1, x_2, x_3\} \in [0,1]^3$ *to the considered system satisfies:* $x_1 = 1, x_2 = 0, x_3 \geq 0.4$.

The converse of Theorem 6 can be proved in the case of a linearly ordered BL-algebra with cancelation. Both statements provide criterion for a maximal solution.

Theorem 7
Let $\mathcal{L}$ *be linearly ordered and the cancelation law holds. Assume that system (1) is solvable. Then* $\bar{\mathbf{x}}$ *is a maximal solution to (1) if and only if*

(i) $\bar{\mathbf{x}}$ *can be represented in the form* $\bar{\mathbf{x}} = \bigwedge_{i=1}^{n} \mathbf{x}^i$ *where each* $\mathbf{x}^i$ *is a suitable maximal solution to the* i*-th equation and*

(ii) $\bigwedge_{i=1}^{n} \mathbf{x}^i \geq \check{\mathbf{x}}$ *holds true.*

PROOF: Due to the fact that the sufficient part of out criterion has been proved in Theorem 6, we will prove the necessary part only. Assume that $\bar{\mathbf{x}}$ is a maximal solution to (1) which implies that $\bar{\mathbf{x}} \geq \check{\mathbf{x}}$ and $\bar{\mathbf{x}}$ is a solution to each i-th equation in (1).

By Lemma 2, for each $i = 1, \ldots n$, the condition $b_i \geq \bigwedge_{j=1}^{m} \neg a_{ij}$ holds and therefore, by Proposition 2, each solution of the i-th equation is less or equal than a respective maximal solution. Let us denote $\mathbf{x}^i$ a maximal solution to the i-th equation such that $\mathbf{x}^i \geq \bar{\mathbf{x}}$. Then $\bigwedge_{i=1}^{n} \mathbf{x}^i \geq \bar{\mathbf{x}}$ and by the transitivity, $\bigwedge_{i=1}^{n} \mathbf{x}^i \geq \check{\mathbf{x}}$. By Theorem 6, $\bigwedge_{i=1}^{n} \mathbf{x}^i$ is a maximal solution to (1) and therefore, $\bigwedge_{i=1}^{n} \mathbf{x}^i = \bar{\mathbf{x}}$. □

These two general theorems being combined with different sufficient conditions of solvability of a single equation proved in Section 3 imply a series of consequences which characterize systems with non-trivial solutions. We will not consider all possible combinations of what we have discovered as sufficient conditions of solvability of a single equation. This is a technical exercise that can be left to the reader. However, we will present some particular cases where a set of solutions contains a single element.

Proposition 4
Let $\mathcal{L}$ *be linearly ordered and for each* $i = 1, \ldots n$,

$$b_i = \bigwedge_{j=1}^{m} \neg a_{ij} \tag{8}$$

so that there exists $j_i : 1 \leq j_i \leq m$, $b_i = \neg a_{ij_i}$. *Let moreover,* $n \geq m$ *and each index* $1, \ldots, m$ *occurs among the indexes* $j_1, \ldots, j_n$. *Then system (1) has the only one solution* $\mathbf{x} = (0, \ldots, 0)$.

PROOF: Let us construct the vector $\check{\mathbf{x}} = (\check{x}_j)$:

$$\check{x}_j = \bigvee_{i=1}^{n} (a_{ij} * b_i) = \bigvee_{i=1}^{n} (a_{ij} * \bigwedge_{l=1}^{m} \neg a_{il}) \leq \bigvee_{i=1}^{n} (a_{ij} * \neg a_{ij}) = 0$$

where $j = 1, \ldots, m$.

According to Theorem 3, $\mathbf{x}_i = (1, \ldots \underbrace{0}_{j_i} \ldots, 1)$ is a maximal solution of the i-th equation in (1), $i = 1, \ldots n$. This fact together with the assumption on permutation leads to

$$\bigwedge_{i=1}^{n} \mathbf{x}_i = (0, \ldots, 0).$$

By Theorem 6, we conclude that $(0, \ldots, 0)$ is the only solution to (1). $\square$

If all but last conditions of Proposition 4 are fulfilled then system (1) has non-trivial maximal solutions. They are characterized in the proposition given below.

Proposition 5
Let $\mathcal{L}$ be linearly ordered and for each $i = 1, \ldots n$, (8) holds true. Let moreover, there exist k, $1 \leq k \leq m$, such that for all $i = 1, \ldots, n$,

$$a_{ik} \neq \bigwedge_{j=1}^{m} \neg a_{ij}.$$

Then each maximal solution contains 1 as the k-th coordinate.

PROOF: By Proposition 3, the vector $\check{\mathbf{x}} = (0, \ldots, 0)$ is a solution. By (8), for each i there exists $j_i : 1 \leq j_i \leq m$, such that $b_i = \neg a_{ij_i}$. Denote by J_0 the set of all those indices j_i having this property for at least one $i = 1, \ldots, n$ and note that by the assumption, $k \notin J_0$. According to Theorem 3, for each $i = 1, \ldots, n$, each maximal solution to the i-th equation in system (1) has 1 on the k-th place. Therefore, their minimum has 1 on the k-th place as well.

$$\bigwedge_{i=1}^{n} \mathbf{x}_i = (\ldots, \underbrace{1}_{k}, \ldots).$$

By Theorem 6, $\bigwedge_{i=1}^{n} \mathbf{x}_i$ is a maximal solution to (1) which has the property required by our proposition. $\square$

Remark 5
In Proposition 5 all other x_j, $j \neq k$, are equal to one or zero.

The following proposition is a little bit tricky and combines Theorem 3, Theorem 5 and Theorem 6.

Proposition 6
Assume that $\mathcal{L}$ is linearly ordered and the cancelation law holds. Let there exist $l \in \{1 \ldots, n\}$ such that $b_l > \neg a_{lk}$ holds true for one element in k-th column of l-th row. For all other row indices $i = 1, \ldots n$, different from l, assume that (8) is fulfilled, so that there exists $j_i : 1 \leq j_i \leq m$, $b_i = \neg a_{ij_i}$. Let moreover, $n \geq m$ and each column index from $\{1, \ldots, m\} \setminus \{k\}$ occurs among the row indexes $\{j_1, \ldots, j_n\} \setminus \{l\}$. Then system (1) has only one solution $\check{\mathbf{x}} = (0, \ldots, \underbrace{a_{lk} * b_l}_{k}, \ldots, 0)$.

PROOF: First, we verify that vector $\check{\mathbf{x}} = (0, \ldots, a_{lk} * b_l, \ldots, 0)$ is a solution. By a direct computation, for $j \neq k$ we obtain

$$\check{x}_j = \bigvee_{i=1}^{n} (a_{ij} * b_i) = \bigvee_{i=1, i \neq l}^{n} (a_{ij} * \bigwedge_{p=1}^{m} \neg a_{ip}) \vee (a_{lj} * b_l) \leq \bigvee_{i=1, i \neq l}^{n} (a_{ij} * \neg a_{ij}) \vee (a_{lj} * \neg a_{lj}) = 0$$

and

$$\check{x}_k = \bigvee_{i=1}^{n} (a_{ik} * b_i) = \bigvee_{i=1, i \neq l}^{n} (a_{ij} * \bigwedge_{p=1}^{m} \neg a_{ip}) \vee (a_{lk} * b_l) = a_{lk} * b_l.$$

By Theorem 5, Proposition 4 and Theorem 6, $\check{\mathbf{x}}$ is a maximal solution. Therefore, it is the only one solution to (1). □

Conclusions

We have shown how systems of linear-like equations in a semilinear space of vectors can emerge from systems of fuzzy relation equations. We have considered the problem of solvability of a system of equations with inf-$\rightarrow$ composition determined by a residual (with respect to some homomorphism) mapping over semilinear spaces. We mainly concentrated on characterization of maximal solutions because of the following facts:

1. solvability of a system is equivalent to existence of the least solution to it,
2. a set of solutions of a system is a $\wedge$-semi-lattice.

We first described full sets of maximal solutions of single equations in the case of a finite universe and different assumptions on an underlying algebra. Depending on the ordering of a support set we obtained either necessary or sufficient conditions, or criteria of being a maximal solution. Then we showed how a maximal solution of a system can be obtained from maximal solutions of single equations. We considered examples where full sets of solutions have been obtained as well.

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