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Abstract: In this paper, we introduce a special algebra called EQ-algebra which has three binary operations (meet, product, fuzzy equality) and a top element. The fuzzy equality is reflexive, symmetric and transitive with respect to the product. EQ-algebra is a natural algebra proposed as an algebra of truth values on the basis of which the fuzzy type theory (a higher-order fuzzy logic) should be developed. Till now, truth values in fuzzy type theory are supposed to form a special residuated lattice. However, since fuzzy equality is a derived operation in residuated lattice, it is not so natural for fuzzy type theory as the EQ-algebra.

Keywords: Residuated lattice, fuzzy equality, fuzzy logic, fuzzy type theory, higher-order fuzzy logic.

1 Introduction

A crucial question for every many-valued logic is, what should be structure of its truth values. It is generally accepted that in fuzzy logic, it should be a residuated lattice, possibly fulfilling some additional properties. On the basis of that, we may now distinguish various kinds of formal fuzzy logics. Most important among them seem to be Łukasiewicz, BL-, MTL-, IMTL- and ŁΠ-fuzzy logics (see [14] for the discussion about proper fuzzy logic). Recall that all these logics have propositional as well as predicate version and enjoy the completeness property. A natural question raises, whether we can introduce also a higher-order fuzzy logic as a counterpart to classical higher-order logic. The latter is called *type theory* (see [1]).

The answer to the above question is positive and the *fuzzy type theory* (FTT) has indeed been introduced in [13]. The algebra of truth values considered there is the IMTL_Δ-algebra which is a residuated lattice fulfilling prelinearity and double negation property and, moreover, it is endowed by the additional operation of Baaz delta Δ (a special unary operation keeping 1 and sending all the other truth values to 0). This is one of the common algebras considered as algebras for truth values for various kinds of fuzzy logic. In [12], other kinds of FTT have been introduced, namely those where the algebra of truth values is one of $\text{Łukasiewicz}_\Delta$, BL_Δ or ŁΠ-algebra. All these algebras have been extensively studied in [3–6, 8] and elsewhere.

However, the basic connective in FTT is a fuzzy equality since it is developed as a generalization of the elegant classical formal system originated by A. Church and L. Henkin (see [1, 9] and the citations therein). Therefore, we introduce in this paper a special algebra that we will call EQ-algebra and that reflects directly the syntax of FTT.

2 EQ-algebras

Definition 1
EQ-algebra is the algebra

$$\mathcal{L} = \langle L, \wedge, \otimes, \sim, \mathbf{1}\rangle \tag{1}$$

of type (2, 2, 2, 0) where for all $a, b, c \in L$*:*

(E1) $\langle L, \wedge, \mathbf{1}\rangle$ *is a commutative idempotent monoid (i.e.* $\wedge$*-semilattice with top element* $\mathbf{1}$*). We put* $a \leq b$ *iff* $a \wedge b = a$*, as usual.*

(E2) $\langle L, \otimes, \mathbf{1}\rangle$ *is a (commutative) monoid and* $\otimes$ *is isotone w.r.t.* $\leq$*.*

(E3) $a \sim a = \mathbf{1}$ *(reflexivity)*

(E4) $a \sim b = b \sim a$ *(symmetry)*

(E5)

(a) $(id(a') \sim c) \otimes (a' \sim a) \leq (id(a) \sim c)$

(b) $((a' \wedge b) \sim c) \otimes (a' \sim a) \leq ((a \wedge b) \sim c)$

(substitution)

(E6) If $c' \leq c$ then

(a) $(a \wedge c') \sim a \leq (a \wedge c) \sim a$

(b) $(a \otimes c) \sim c \leq (a \wedge c') \sim c'$

(monotonicity)

(E7) $a \wedge b \leq a \sim b$ *(boundedness)*

The operation $\wedge$ is called meet (infimum), $\otimes$ is called product and $\sim$ is a fuzzy equality.

Clearly, $\leq$ is the classical partial order. We will also put $\tilde{a} = a \sim \mathbf{1}$ and

$$a \to b = (a \wedge b) \sim a. \tag{2}$$

The substitution axiom can be seen also as a special form of the extensionality (see [8] and elsewhere).

Lemma 1
The following transitivity properties hold in EQ-algebra:

(a) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$,

(b) $(a \to b) \otimes (b \to c) \leq a \to c$.

PROOF: (a) is just (E5)(a) for $a' = b$ by (E4).
(b) Note that by (E6), $(a \wedge b \wedge c) \sim a \leq (a \wedge c) \sim a$. Furthermore, by (E5) we have

$$(a \wedge (b \wedge c)) \sim a \geq ((a \wedge b) \sim a) \otimes ((b \wedge c) \sim b).$$

From this follows

$$((a \wedge b) \sim a) \otimes ((b \wedge c) \sim b) \leq (a \wedge (b \wedge c)) \sim a \leq (a \wedge c) \sim a$$

which is (b). □

Remark 1
The substitution axiom (E5) can be strengthened as follows: Let H be a set of functions on L. If $h \in H$ then we will write its arity as $Ar(h)$. Then

$$(h(a'_1, \ldots, a'_n) \sim c) \otimes (a'_1 \sim a_1) \cdots \otimes (a'_n \sim a_n) \leq (h(a_1, \ldots, a_n) \sim c) \quad \text{(E5')}$$

holds all functions $h : L^n \longrightarrow L$ where $h \in H$ and $Ar(h) = n$. We may even weaken (E5') by admitting exponents, i.e. to assume existence of $m_1, \ldots, m_n > 0$ such that

$$(h(a'_1, \ldots, a'_n) \sim c) \otimes (a'_1 \sim a_1)^{m_1} \cdots \otimes (a'_n \sim a_n)^{m_n} \leq (h(a_1, \ldots, a_n) \sim c). \quad \text{(E5'')}$$

Note also that axiom (E6), in fact, expresses isotonicity of $\to$ w.r.t. the second variable and antitonicity of $\to$ w.r.t. the first variable.

Lemma 2
The following holds for all $a, b, c \in L$ in every EQ-algebra.

(a) $a \otimes b \leq a$, $a \otimes b \leq a \wedge b$, $c \otimes (a \wedge b) \leq (c \otimes a) \wedge (c \otimes b)$.

(b) $a \sim b \leq a \to b$ *and* $a \to a = \mathbf{1}$.

(c) $(a \to b) \otimes (b \to a) \leq (a \sim b)$.

(d) $a = b$ *implies* $a \sim b = \mathbf{1}$.

(e) *If* $a \leq b$ *then* $a \to b = \mathbf{1}$, $a \sim b = b \to a$ *and* $\tilde{a} \leq \tilde{b}$.

(f) $a \leq \tilde{a}$ *and* $\tilde{\mathbf{1}} = \mathbf{1}$.

(g) $\tilde{a} = \mathbf{1} \to a$ *and* $a \to \mathbf{1} = \mathbf{1}$.

(h) $a \otimes (a \sim b) \leq \tilde{b}$.

(i) $a \otimes b \leq \tilde{a} \otimes \tilde{b} \leq a \sim b$.

(j) $(a \sim a') \otimes ((a' \wedge b) \to c) \leq ((a \wedge b) \to c)$ *and* $(a' \to c) \otimes (a' \sim a) \leq (a \to c)$.

(k) $b \leq \tilde{b} \leq a \to b$.

(l) $((a \wedge b) \sim (c \wedge d)) \otimes (a \sim a') \otimes (b \sim b') \otimes (c \sim c') \otimes (d \sim d') \leq (a' \wedge b') \sim (c' \wedge d')$.

PROOF: (a) Immediately from (E1) and (E2) by isotonicity of $\otimes$.
(b) By (E5)(b) and (E3) we get

$$a \to b = (a \wedge b) \sim a \geq ((a \wedge a) \sim a) \otimes (a \sim b) = a \sim b.$$

Further, $a \to a = (a \wedge a) \sim a = \mathbf{1}$.
(c) By definition of $\to$ and Lemma 1(a) we get
$(a \to b) \otimes (b \to a) = ((a \wedge b) \sim a) \otimes ((a \wedge b) \sim b) \leq (a \sim b)$.
(d) If $a = b$ then, obviously, $a \sim b = a \sim a = \mathbf{1}$.
(e) We have $a \wedge b = a$, so that $(a \wedge b) \sim a = a \sim a = \mathbf{1}$ and $a \sim b = (a \wedge b) \sim b$.
(f) Immediately from (E7) by the properties of $\otimes$.
(g) By definition of $\to$, we get $\tilde{a} = a \sim \mathbf{1} = (\mathbf{1} \wedge a) \sim \mathbf{1} = \mathbf{1} \to a$ and $a \to \mathbf{1} = (a \wedge \mathbf{1}) \sim a = a \sim a = \mathbf{1}$.
(h) and (i) follow immediately from Lemma 1(a), (f) and monotonicity of $\otimes$.
(j) By (E5)(b) we have
$((a' \wedge (b \wedge c) \sim c) \otimes (a \sim a') \leq ((a \wedge (b \wedge c) \sim c)$.
(k) By (f) and (E5)(b), we get
$b \leq \mathbf{1} \to b = (\mathbf{1} \wedge b) \sim \mathbf{1} \leq (a \wedge b) \sim a = a \to b$.
(l) is a consequence of (E6)(b) and (E2). □

Example 1
The algebra

$$\mathcal{L} = \langle [0, 1], \wedge, \otimes, \sim, 1 \rangle$$

where $\otimes$ is some Yager t-norm weaker than Łukasiewicz product and

$$x \sim y = 1 - |f(x) - f(y)|$$

for some non-decreasing function $f : [0, 1] \longrightarrow [0, 1]$ is an EQ-algebra. We have $\tilde{x} = f(x)$.

Let us put put

$$a \leftrightarrow b = (a \to b) \wedge (b \to a),$$
$$a \hat{\Leftrightarrow} b = (a \to b) \otimes (b \to a).$$

Example 2
Let $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \to, \mathbf{0}, \mathbf{1}\rangle$ be a residuated lattice. Then both $\mathcal{L} = \langle L, \wedge, \otimes, \leftrightarrow, \mathbf{1}\rangle$ as well as $\mathcal{L} = \langle L, \wedge, \otimes, \hat{\Leftrightarrow}, \mathbf{1}\rangle$ are EQ-algebras. Moreover, the substitution axiom (E5') holds for $h \in \{\wedge, \otimes, \leftrightarrow, \hat{\Leftrightarrow}\}$ in both EQ-algebras.

Lemma 3
Let $a \leq b \to c$. Then

$$a \otimes b \leq \tilde{c}.$$

PROOF: This follows from the assumption and Lemma 2(h). □

The following two fuzzy equalities have been introduced in [7].

Lemma 4
Let $\sim$ be a fuzzy equality and put

$$a \overset{\wedge}{\sim}_c b = c \wedge (a \sim b),$$
$$a \overset{\otimes}{\sim}_c b = c \otimes (a \sim b)$$

and $a \overset{\wedge}{\sim}_c a = a \overset{\otimes}{\sim}_c a = \mathbf{1}$ for arbitrary $a, c \in L$. Then both $\overset{\wedge}{\sim}_c$ as well as $\overset{\otimes}{\sim}_c$ are fuzzy equalities fulfilling (E3)–(E6).

PROOF: This follows from the properties of $\sim$, $\wedge$ and $\otimes$. □

The following lemma characterizes a compatibility of $\sim$ with the ordering $\leq$, namely that if "distance" of elements increases (in the sense of $\leq$) then the degree of their equality decreases.

Lemma 5
If $a \leq b \leq c$ then $c \sim a \leq c \sim b$ as well as $a \sim c \leq a \sim b$.

PROOF: Since $a \leq b$, we immediately get from (E6) that $c \sim a = (a \wedge c) \sim c \leq (b \wedge c) \sim c = c \sim b$ as well as $a \sim c = (a \wedge c) \sim c \leq (a \wedge b) \sim b = a \sim b$. □

Remark 2
Some studies of Höhle (see, e.g. [11]) suggest to consider, in a sense, a more general algebra with the reflexivity axiom (E3) replaced by

$$a \sim b \leq (a \sim a) \wedge (b \sim b) \qquad (3)$$

and the transitivity axiom (E6)(a) (cf. Lemma 1(a)) replaced by local transitivity

$$(a \sim b) \otimes ((b \sim b) \to (b \sim c)) \leq a \sim c. \qquad (4)$$

Local transitivity is stronger than the transitivity in Lemma 1(a).

The EQ-algebra can be seen as a set endowed with a classical partial order with classical equality and a top element, and a fuzzy equality together with a fuzzy ordering, i.e. a structure

$$\langle L, =, \leq, \sim, \lesssim, \mathbf{1}\rangle \qquad (5)$$

where $\sim, \lesssim \in L^{L\times L}$. Indeed, $\sim$ is clearly a fuzzy equality by (E3), (E4) and (E6) so that $=$ is, by Lemma 2(d), its special case.

Furthermore, put $a \lesssim b = a \to b$ (this notation is introduced only for transparency). From Lemma 2(e) we get: $a \leq a$ implies $(a \lesssim a) = \mathbf{1}$, i.e. $\lesssim$ is reflexive. Furthermore, by Lemma 2(c) we get

$$(a \lesssim b) \otimes (b \lesssim a) \leq a \sim b,$$

i.e. $\lesssim$ is antisymmetric w.r.t. $\sim$. Finally, by Lemma 1(b)

$$(a \lesssim b) \otimes (b \lesssim c) \leq (a \lesssim c), \qquad (6)$$

i.e. $\lesssim$ is also transitive. Therefore, the fuzzy relation $\lesssim$ in (5) is a fuzzy ordering. Note that when writing (2) as

$$a \lesssim b = (a \wedge b) \sim a \qquad (7)$$

and understanding $\sim$ as a fuzzy equality then (7) becomes just a generalization of the classical definition $a \leq b$ iff $a \wedge b = a$.

Let $\mathcal{L}$ contain also the bottom element $\mathbf{0}$. Then we may put

$$\neg a = a \sim \mathbf{0}, \qquad a \in L. \qquad (8)$$

Lemma 6
(a) $\neg\mathbf{1} = \tilde{\mathbf{0}}$, $\neg\mathbf{0} = \mathbf{1}$.

(b) $\mathbf{0} \to a = \mathbf{1}$, $\neg a = a \to \mathbf{0}$.

(c) If $a \leq b$ then $\neg b \leq \neg a$.

(d) $\neg\tilde{\mathbf{0}} = \neg\neg\mathbf{1} \leq \mathbf{1}$

(e) $a \otimes \neg a \leq \tilde{\mathbf{0}}$, $\tilde{a} \otimes \tilde{\mathbf{0}} \leq \neg a$, $\neg a \otimes \tilde{\mathbf{0}} \leq \tilde{a}$ *and* $a \otimes \mathbf{0} = \mathbf{0}$.

(f) $\neg a \otimes \neg b \leq a \to b$.

PROOF: (a) is obvious, (b) follows from Lemma 2(e), (c) follows from Lemma 5, (d) follows from (c) and (e) follows from Lemma 2(h), Lemma 1(a) and Lemma 2(a). (f) By (E5)(b) we have $((\mathbf{0} \wedge a) \sim a) \otimes (b \sim \mathbf{0}) \leq (a \wedge b) \sim a = a \to b$. □

Definition 2
(i) EQ-algebra is spanned *if*

(E8) $\tilde{\mathbf{0}} = \mathbf{0}$.

(ii) EQ-algebra is good *if for all $a \in L$,*

(E9) $a \sim \mathbf{1} = a$.

(iii) *EQ-algebra is* separated *if for all* $a, b \in L$,

(E10) $a \sim b = \mathbf{1}$ *implies* $a = b$.

(iv) *EQ-algebra is* residuated *if for all* $a, b, c \in L$,

(E11) $(a \otimes b) \wedge c = a \otimes b$ *iff* $a \wedge ((b \wedge c) \sim b) = a$.

Obviously, if the EQ-algebra is good then it is spanned but note vice-versa. In [13], the property (E10) of the fuzzy equality $\sim$ is called 1-faithfulness. The term "separated" has been earlier introduced by U. Höhle [10]. Clearly, (E11) can be written in a classical way as $a \otimes b \leq c$ iff $a \leq b \to c$.

In a good EQ-algebra, many properties from Lemmas 2 and 6 become the standard properties known from the theory of residuated lattices.

Lemma 7
(a) In every good EQ-algebra $a \leftrightarrow \mathbf{1} = a \,\hat{\Leftrightarrow}\, \mathbf{1} = a$.

(b) A good EQ-algebra $\mathcal{L}$ *is residuated if*

$$(a \otimes b) \wedge c = a \otimes b \quad \textit{implies} \quad a \wedge ((b \wedge c) \sim b) = a$$

for all $a, b, c \in L$.

(c) An EQ-algebra $\mathcal{L}$ *is good iff*

$$a \otimes (a \sim b) \leq b \qquad (9)$$

for all $a, b \in L$.

(d) If a good EQ-algebra fulfils

$$(a \sim a') \leftrightarrow (b \sim b') \leq (a \sim b) \leftrightarrow (a' \sim b') \qquad (10)$$

for all $a, b, a', b' \in L$ *then* $\sim = \leftrightarrow$.

PROOF: (a) This follows immediately from Lemma 2(g).
(b) follows from Lemma 3.
(c) If (9) holds then $\mathbf{1} \otimes (\mathbf{1} \sim b) \leq b$ holds for all $b \in L$, i.e. $\tilde{b} = b$ by Lemma 2(f). The converse implication follows from Lemma 2(h).
(d) Put $a' = b' = \mathbf{1}$. Then by (10) we obtain $(a \sim \mathbf{1}) \leftrightarrow (b \sim \mathbf{1}) = a \leftrightarrow b \leq (a \sim b) \leftrightarrow (\mathbf{1} \sim \mathbf{1}) = a \sim b$. The equality follows from Lemma 8(b). □

The following lemma characterizes induced fuzzy relations $\leftrightarrow$ and $\hat{\Leftrightarrow}$.

Lemma 8
The following holds in every EQ-algebra $\mathcal{L}$*:*

(a) $(a \wedge b) \leftrightarrow a = (a \wedge b) \,\hat{\Leftrightarrow}\, a = a \to b$.

(b) $a \,\hat{\Leftrightarrow}\, b \leq a \sim b \leq a \leftrightarrow b$.

(c) Both $\leftrightarrow$ *as well as* $\hat{\Leftrightarrow}$ *are fuzzy relations fulfilling (E3)–(E5)(a) and (E6)–(E7).*

(d) If $\mathcal{L}$ *is residuated then* $\leftrightarrow$ *as well as* $\hat{\Leftrightarrow}$ *are fuzzy equalities.*

(e) If $\mathcal{L}$ *is linearly ordered then* $a \leftrightarrow b = a \,\hat{\Leftrightarrow}\, b = a \sim b$.

PROOF: (a) $(a \wedge b) \leftrightarrow a = ((a \wedge b) \to a) \wedge (a \to (a \wedge b))$ and $(a \wedge b) \to a = \mathbf{1}$. The second part is similar.
(b) follows from Lemma 2(c) and (b) by symmetry of $\sim$ and the properties of $\wedge$.
(c) (E3) follows from Lemma 2(b); (E4) is obvious.
(E5)(a) is obtained from the facts that $a \leftrightarrow b \leq a \to b$ and Lemma 1(b) by the properties of $\wedge$ and $\otimes$. Note that this is transitivity of both fuzzy relations in the sense of Lemma 1(a).
(E6) follows from (a).
(E7) follows from Lemma 2(k) and the properties of $\otimes$.
(d) It remains to show (E5)(b). By the properties of $\otimes$ and $\wedge$, it is sufficient to demonstrate that

$$(c \to (a' \wedge b)) \otimes (a' \to a) \leq (c \to (a \wedge b))$$

as well as

$$((a' \wedge b) \to c) \otimes (a \to a') \leq ((a \wedge b) \to c).$$

These inequalities are a consequence of residuation and Lemmas 2(a) and 7(c).
(e) If $a \leq b$ then $a \sim b = b \to a$ and $a \to b = \mathbf{1}$ by Lemma 2(e). □

Lemma 9
Let $\mathcal{L}$ *be a residuated EQ-algebra. Then it is good and separated.*

PROOF: Since $a \to b \leq a \to b$ then, by residuation, $a \otimes (a \to b) \leq b$, i.e. a residuated EQ-algebra is good by Lemma 7(c). Furthermore, let $a \sim b = \mathbf{1}$. Then Lemma 8(b) implies that $\mathbf{1} \leq (a \to b) \wedge (b \to a)$, so that $a \leq b$ as well as $b \leq a$ by residuation. □

The EQ-algebra is *complete* if it is a complete $\wedge$-semilattice. This immediately implies (see [2]) that, since it contains a top element, a complete EQ-algebra is at the same time a complete lattice.

Lemma 10
The following holds in every complete EQ-algebra:

(a) $a \to \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I} (a \to b_i)$.

(b) $\bigvee_{i \in I} (a_i \to b) \leq (\bigwedge_{i \in I} a_i \to b)$.

(c) $\bigvee_{i \in I} ((a_i \to b_i) \otimes (a_i \to \bigwedge_{i \in I} a_i) \otimes (b_i \to \bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} a_i \to \bigwedge_{i \in I} b_i$.

PROOF: All inequalities are consequence of the properties of sup and inf, (E6) and Lemma 2(l). □

Definition 3
A lattice EQ-algebra *(ℓEQ-algebra) is an EQ-algebra that is a lattice and, moreover, the following additional substitution axiom holds:*

(E12) $((a' \vee b) \sim c) \otimes (a' \sim a) \leq ((a \vee b) \sim c)$.

Obviously, a complete EQ-algebra is a complete ℓEQ-algebra. A residuated ℓEQ-algebra is a residuated lattice.

Lemma 11
If $\mathcal{L}$ is an ℓEQ-algebra then

$$a \to b = (a \vee b) \sim b. \tag{11}$$

PROOF: By (E5), we get

$$(a \wedge (a \vee b) \sim a) \otimes (a \vee b \sim b) \leq (a \wedge b) \sim a = a \to b.$$

Conversely, by (E12) we get

$$(b \vee (a \wedge b) \sim b) \otimes (a \wedge b \sim a) = a \to b \leq (a \vee b) \sim b.$$

□

It follows from this lemma that the fuzzy ordering $\lesssim$ in the ℓEQ-algebra can be defined alternatively, either by $a \lesssim b = (a \wedge b) \sim a$ or by $a \lesssim b = (a \vee b) \sim b$, just as classical ordering in the lattice.

Lemma 12
Let $\mathcal{L}$ be an ℓEQ-algebra. Then

(a) $((b \wedge c) \sim a) \otimes ((b \vee c) \to b) \leq c \sim a$.

(b) $((a \wedge (b \vee c)) \sim d) \otimes (a \sim b) \leq b \sim d$.

PROOF: This follows immediately from (E5)(b) and the absorption law. □

3 Conclusion

In this paper, we have introduced a special algebra called EQ-algebra that will serve as the algebra of truth values for fuzzy type theory where the main connective is a fuzzy equality. Recall that till now, FTT has been developed on the basis of four kinds of residuated lattices which, however, are special kinds of our EQ-algebra. Thus, FTT developed on the basis of the latter will be the most general.

We have introduced the basic definition and several special kinds of EQ-algebras and studied their basic properties. Much algebraic study must still be done to be able to fulfil the above task — to develop FTT based on EQ-algebra.

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