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Abstract

Fuzzy transform is a known technique that belongs into the area of fuzzy approximations. This method consists of the direct fuzzy transform and the inverse fuzzy transform.

Untill now this method was used and analyzed only for the continuous functions. In this contribution we generalize this method for the case of discontinuous functions and we prove its convergent properties.

Keywords: Fuzzy transform, Fuzzy Approximation, Convergence.

1 Introduction

The fuzzy transform (F-transform) is one of the approximation methods. This method has been developed by I. Perfilieva and presented in [1, 2]. We will describe F-transform and generalize this method for functions from the L_1 - space. Finally we will study pointwise convergence in a points of continuity and discontinuity.

2 Fuzzy Transform

We take a bounded interval $[a, b]$ as a universe. Let us introduce fuzzy sets which are subsets of the universe $[a, b]$ and which form fuzzy partition of the universe.

Definition 1
Let $x_1 < x_2 < \ldots < x_n$ be nodes on $[a, b]$ such that $x_1 = a$ and $x_n = b$ and $n \geq 2$. We say that fuzzy sets $A_1, \ldots, A_n$, identified with their membership functions $A_1, \ldots, A_n$ defined on $[a, b]$ form a fuzzy partition of $[a, b]$ if they fulfil the following conditions for all $k = 1, \ldots, n$:

- $A_k : [a, b] \to [0, 1], A_k(x_k) = 1$
- $A_k(x) = 0$ *if* $x \notin (x_{k-1}, x_{k+1})$ *where for the uniformity of denotation, we put* $x_0 = a$ *and* $x_{n+1} = b$
- A_k *is continuous*
- A_k, $k = 2, \ldots, n$ *strictly increases on* $[x_{k-1}, x_k]$ *and* A_k, $k = 1, \ldots, n-1$ *strictly decreases on* $[x_k, x_{k+1}]$
- $\sum_{k=1}^{n} A_k(x) = 1$ *for all* $x \in [a, b]$

The membership functions $A_1, A_2, \ldots, A_n$ are called basic functions.

We say that fuzzy partition is uniform if the nodes $x_1, \ldots, x_n$, $n \geq 3$ are equidistant.

For a uniform fuzzy partition, the following properties hold:

- $A_k(x_k - x) = A_k(x_k + x)$, for all $x \in [0, h]$, where $k = 2, \ldots, n-1$, $n > 2$ and $h = (b-a)/(n-1)$
- $A_{k+1}(x) = A_k(x - h)$, for all $x \in [a + h, b]$, where $k = 2, \ldots, n-2$, $n > 2$ and $h = (b-a)/(n-1)$.

In the pictures 2 and 2, there are examples of uniform fuzzy partitions.

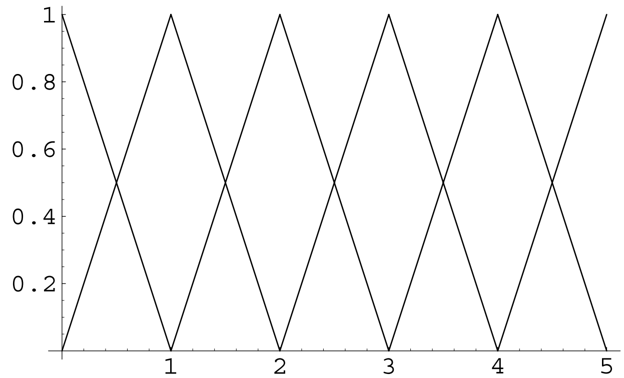

Figure 1: Example of fuzzy partition with triangular basic functions

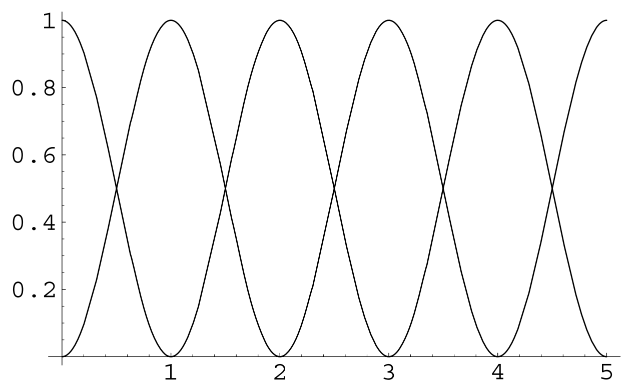

Figure 2: Example fuzzy partition with sinusoidal basic functions

Remark 1
We will characterize the fuzzy partition by parameter

$$h(n) = \max_{k=1,\ldots,n-1} (x_{k+1} - x_k).$$

We will suppose that $h(n) \to 0$ *for* $n \to \infty$.

F-transform is created by the direct and the inverse F-transform.

Definition 2
Let $A_1, \ldots, A_n$ *be basic functions which form a fuzzy partition and* $f(x)$ *be any function from* $L_1(a, b)$. *We say that the n-tuple of real numbers* $[F_1, F_2, \ldots, F_n]$ *given by*

$$F_k = \frac{\int_a^b f(x) A_k(x) dx}{\int_a^b A_k(x) dx}, \qquad k = 1, \ldots, n.$$

is the direct F-transform of function f *with respect to* $A_1, \ldots, A_n$.

By the direct F-transform we get approximately values in the nodes of fuzzy partition. Now we will define the inverse F-transform.

Definition 3
Let $[F_1, F_2, \ldots, F_n]$ be the direct F-transform with respect to the fuzzy partition $A_1, A_2, \ldots, A_n$. The function

$$f_{F,n}(x) = \sum_{k=1}^{n} F_k A_k(x)$$

is called the inverse F-transform.

By the inverse F-transform we get the approximation of the original function.

Now we introduce the basic convergence theorem for continuous functions.

Theorem 1
Let f be a continuous function on the bounded interval $[a,b]$. Let $\{A_1^{(n)}, \ldots, A_n^{(n)}\}_{n=1}^{\infty}$ be any sequence of fuzzy partitions of $[a,b]$ with $h(n) \to 0$. Then the sequence $\{f_{F,n}\}_{n=1}^{\infty}$ of the inverse F - transforms, each with respect to the given n-tuple $A_1^{(n)}, \ldots, A_n^{(n)}$, uniformly converges to the function f.

PROOF: We will follow the proof from I. Perfilieva published in [1]. Let $\varepsilon > 0$. Note that the function f is uniformly continuous on $[a,b]$; i.e., there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in [a,b]$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $A_1^{(n)}, \ldots, A_n^{(n)}$ be a fuzzy partition with $h(n) < \delta/2$. Denote $F_1, \ldots, F_n$ as the corresponding components of the F-transform and $f_{F,n}$ as the inverse F-transform of f with respect to the fuzzy partition. Let t be any point in $[a,b]$ and let x_k, $k \in \{1, 2, \ldots, n-1\}$, be the node of the partition such that $t \in [x_k, x_{k+1}]$. Then

$$|f(t) - F_k| = \left| f(t) - \frac{\int_{x_{k-1}}^{x_{k+1}} f(x) A_k^{(n)}(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k^{(n)}(x)dx} \right| \leq \frac{\int_{x_{k-1}}^{x_{k+1}} |f(x) - f(t)| A_k^{(n)}(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k^{(n)}(x)dx} < \varepsilon$$

and analogously,

$$|f(t) - F_{k+1}| < \varepsilon.$$

Hence, by using the fact that only the basic functions A_k and A_{k+1} are not vanished in the point t,

$$|f(t) - f_{F,n}(t)| \leq \sum_{i=k}^{k+1} A_i^{(n)}(t)|f(t) - F_i| < \varepsilon \sum_{i=k}^{k+1} A_i^{(n)}(t) = \varepsilon$$

□

3 Convergence in the Space $L^1(a,b)$

We know that the method of fuzzy transform is well-defined for functions from the space $L^1(a,b)$. So it is natural to ask whether the fuzzy approximation is correct in this space.

Theorem 2
Let (a,b) is a bounded interval and function $f \in L^1(a,b)$. Let $\{A_1^{(n)}, \ldots, A_n^{(n)}\}_{n=1}^{\infty}$ be any sequence of fuzzy partitions of $[a,b]$ with $h(n) \to 0$. Then the sequence $\{f_{F,n}\}_{n=1}^{\infty}$ of the inverse F - transforms, each with respect to the given n-tuple $A_1^{(n)}, \ldots, A_n^{(n)}$, converges to f in the space $L^1(a,b)$.

PROOF: Let $\varepsilon > 0$. Because the space $C([a,b])$ is dense in $L^1(a,b)$ there exists function $\varphi \in C([a,b])$ such that

$$\|f - \varphi\|_1 < \varepsilon/4. \tag{1}$$

By Theorem 1 there exists $n \in \mathbb{N}$ such that

$$\|\varphi_{F,n} - \varphi\|_1 < \varepsilon/4, \tag{2}$$

where $\varphi_{F,n}$ is the inverse F-transform of φ with respect to the fuzzy partition $A_1^{(n)},\dots,A_n^{(n)}$. Let $f_{F,n}$ be the inverse F-transform of f with respect to the fuzzy partition $A_1^{(n)},\dots,A_n^{(n)}$. Then

$$\begin{aligned}\|\varphi_{F,n}-f_{F,n}\|_1 &= \int_a^b \left|\sum_{i=1}^n A_i^{(n)}(t)\frac{\int_{x_{i-1}}^{x_{i+1}}(f-\varphi)A_i^{(n)}dx}{\int_{x_{i-1}}^{x_{i+1}}A_i^{(n)}(x)dx}\right| dt \le \sum_{i=1}^n \int_{x_{i-1}}^{x_{i+1}} |f(x)-\varphi(x)|A_i^{(n)}(x)dx \\ &\le \sum_{i=1}^n \int_{x_{i-1}}^{x_{i+1}} |f(x)-\varphi(x)|dx \le 2\int_a^b |f(x)-\varphi(x)|dx = 2\|f-\varphi\|_1 < \frac{\varepsilon}{2}\end{aligned}$$

From the inequalities (1), (2) and (3) we have

$$\|f-f_{F,n}\|_1 \le \|f-\varphi\|_1 + \|\varphi_{F,n}-\varphi\|_1 + \|\varphi_{F,n}-f_{F,n}\|_1 < \varepsilon.$$

□

4 Pointwise Convergence of F-Transform

In this section we will analyze pointwise convergence of the F-transform method in a points of continuity and discontinuity. In some cases there is important whether the considered point is a node of fuzzy partitions.

4.1 Pointwise Convergence at a Point of Continuity

Theorem 3
Let function f be continuous at a point $t \in [a,b]$. Let $\{A_1^{(n)},\dots,A_n^{(n)}\}_{n=1}^{\infty}$ be any sequence of fuzzy partitions of $[a,b]$ with $h(n)\to 0$. Then the sequence $\{f_{F,n}\}_{n=1}^{\infty}$ of the inverse F - transforms, each with respect to the given n-tuple $A_1^{(n)},\dots,A_n^{(n)}$, converges to the function f at the point t.

PROOF: Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f(x)-f(t)| < \varepsilon$ for all $x \in (t-\delta, t+\delta)$. Let $A_1^{(n)},\dots,A_n^{(n)}$ be a fuzzy partition with $h(n) < \delta/2$. Let $F_1,\dots,F_n$ be the components of the F-transform and $f_{F,n}$ be the inverse F-transform of f with respect to the fuzzy partition. Let $t \in [x_k, x_{k+1}]$ where x_k is the node of the partition. Then

$$|f(t)-F_k| = \left|f(t) - \frac{\int_{x_{k-1}}^{x_{k+1}} f(x)A_k^{(n)}(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k^{(n)}(x)dx}\right| \le \frac{\int_{x_{k-1}}^{x_{k+1}} |f(x)-f(t)|A_k^{(n)}(x)dx}{\int_{x_{k-1}}^{x_{k+1}} A_k^{(n)}(x)dx} < \varepsilon$$

and analogously,

$$|f(t)-F_{k+1}| < \varepsilon.$$

Hence

$$|f(t)-f_{F,n}(t)| \le \sum_{i=k}^{k+1} A_i^{(n)}(t)|f(t)-F_i| < \varepsilon \sum_{i=k}^{k+1} A_i^{(n)}(t) = \varepsilon$$

□

4.2 Pointwise Convergence at a Point of Discontinuity

We will only analyze the case when a function is continuous from the left and right side at a considered point (see Figure **??**).

Theorem 4
Let function f be discontinuous at a point $t \in [a,b]$. Let $f^+(t) = \lim_{x\to t^+} f(x)$ and $f^-(t) = \lim_{x\to t^-} f(x)$. Let $\{f_{F,n}\}_{n=1}^{\infty}$ be an arbitrary sequence of the inverse F-transforms of function f with respect to the sequence of partitions $\{A_1^{(n)}, \ldots, A_n^{(n)}\}_{n=1}^{\infty}$ such that $h(n) \to 0$. Let t be a node of the each fuzzy partition. Then $\liminf_{n\to\infty} f_{F,n}(t)$ and $\limsup_{n\to\infty} f_{F,n}(t)$ lie between $f^+(t)$ and $f^-(t)$.

In addition if the fuzzy partitions are uniform then

$$\lim_{n\to\infty} f_{F,n}(t) = \frac{f^+(t) + f^-(t)}{2}. \tag{3}$$

PROOF: Denote $t - \delta_1^n, t + \delta_2^n$ as neighbouring nodes of the node t where $\delta_1^n, \delta_2^n > 0$ and $\delta_1^n, \delta_2^n \to 0$ for $n \to \infty$. Let

$$\begin{aligned} \eta_1^n &= \int_{t-\delta_1^n}^{t} A_t^{(n)}(x)dx, \\ \eta_2^n &= \int_{t}^{t+\delta_2^n} A_t^{(n)}(x)dx \end{aligned}$$

where $A_t^{(n)}(x)$ is a basic function corresponding to the node t. Clearly $\eta_1^n, \eta_2^n > 0$ and $\eta_1^n, \eta_2^n \to 0$ for $n \to \infty$. Let

$$G_n = \frac{f^-(t)\eta_1^n + f^+(t)\eta_2^n}{\eta_1^n + \eta_2^n}. \tag{4}$$

Then $\liminf_{n\to\infty} G_n$ and $\limsup_{n\to\infty} G_n$ lies between $f^-(t)$ and $f^+(t)$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $|f(x) - f^-(t)| < \varepsilon$ for all $x \in (t-\delta, t)$ and $|f(x) - f^+(t)| < \varepsilon$ for all $x \in (t, t+\delta)$. Choose $n \in \mathbb{N}$ such that $\delta_1^n, \delta_2^n < \delta$. Then

$$|f_{F,n}(t) - G_n| \le \frac{\int_{t-\delta_1^n}^{t} |f(x) - f^-(t)| A_t^{(n)}(x)dx}{\eta_1^n + \eta_2^n} + \frac{\int_{t}^{t+\delta_2^n} |f(x) - f^+(t)| A_t^{(n)}(x)dx}{\eta_1^n + \eta_2^n} \le \varepsilon.$$

Hence

$$\liminf_{n\to\infty} f_{F,n}(t) = \liminf_{n\to\infty} G_n,$$

$$\limsup_{n\to\infty} f_{F,n}(t) = \limsup_{n\to\infty} G_n.$$

If the fuzzy partitions are uniform then by using the properties of a uniform partition

$$G_n = \frac{f^+(t) + f^-(t)}{2},$$

holds true for all n which implies (3). □

Remark 2
Suppose that there exists $C \in [0, +\infty]$ such that $C = \lim_{n\to\infty}(\eta_1^n/\eta_2^n)$. Then by using (4)

$$\lim_{n\to\infty} f_{F,n}(t) = \lim_{n\to\infty} G_n =$$

$$= \begin{cases} \frac{Cf^-(t) + f^+(t)}{C+1}, & C < +\infty, \\ f^-(t), & C = +\infty. \end{cases}$$

If a point of discontinuity is not a node of a uniform fuzzy partition then the convergence need not hold. We will illustrate this situation in a following example.

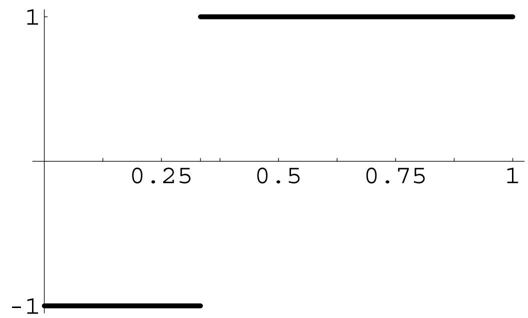

Figure 3: Given function f.

Example We consider the sequence of equidistant partition of the interval $[0,1]$ with discretization parameter $h = 1/2^n$. Let

$$f(x) = \begin{cases} -1, & x \in [0, 1/3), \\ 1, & x \in [1/3, 1]. \end{cases}$$

The graph of this function is in the Figure 4.2.

The point of discontinuity $t = 1/3$ lies between the nodes $i/2^n$ and $(i+1)/2^n$ with

$$i = \frac{[2^n/3]}{2^n},$$

where $[x]$ is the whole part of number x. Then $F_i = \delta^2 - 2\delta$ and $F_{i+1} = 1 - \delta^2$ where

$$\delta = \frac{2^n}{3} - \left[\frac{2^n}{3}\right] = \begin{cases} 1/3, & n - even, \\ 2/3, & n - odd. \end{cases}$$

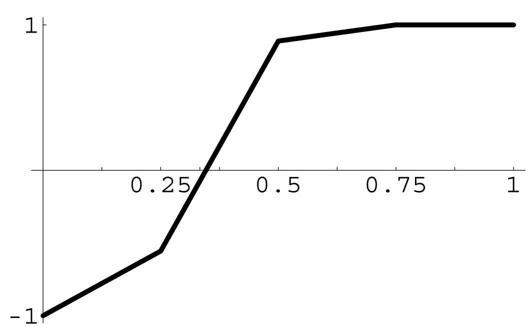

Figure 4: The inverse F-transform $f_{F,2}$ of original function f with respect to the equidistant partition with parameter $h = 1/4$.

Hence

$$f_{F,n}(1/3) = \begin{cases} -2/27, & n - even, \\ 2/27, & n - odd. \end{cases}$$

Therefore $\lim_{n\to\infty} f_{F,n}(1/3)$ does not exist.

5 Conclusion

We have introduced a technique of the direct and the inverse F-transform and have analyzed its properties for the case of continuous and discontinuous functions. It has been shown that the sequence of inverse

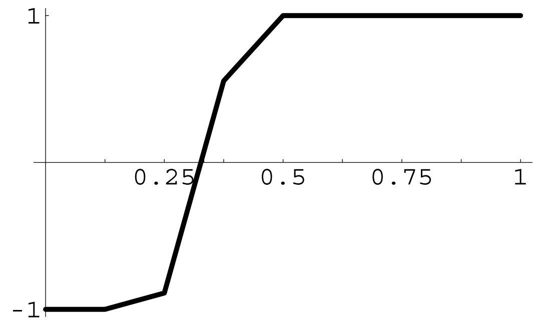

Figure 5: The inverse F-transform $f_{F,3}$ of original function f with respect to the equidistant partition with parameter $h = 1/8$.

of F-transform always converges for uniform partitions. In the case of non-uniform partitions need not converge for piecewise continuous functions.

In the future, we will consider the problem of convergence for sequence of arithmetic wears of inverse F-transforms. We will also investigate the principle difference between Fourier and F-transforms.

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