

Non-commutative first-order EQ-logic

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Outline

- 1 **Motivation**
- 2 **EQ-algebras**
- 3 **Propositional EQ-logics**
 - Basic EQ-logic
 - Extensions
 - Prelinear EQ_{Δ} -logic
- 4 **Predicate EQ-logic**
- 5 **Conclusion**

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How did EQ-logic arise?

- Motivation comes from G. W. Leibniz, L. Wittgenstein and F. P. Ramsey. To develop logic on the basis of identity (equality) as the principle connective.
- Henkin's type theory (higher ordered logic) was developed. [L. Henkin, A theory of propositional types, *Fundamenta Math.*, 52: 323–344, (1963).]
A fully satisfactory logical calculus must be an equational one."
- Classical equality-based logic:
[D. Gries, F. B. Schneider. Equational propositional logic. *Information Processing Letters*, 53:145-152, 1995.]
[G. Tournakis. Mathematical Logic. New York, J.Wiley & Sons, 2008.]

How did EQ-logic arise?

How could fuzzy logic be developed on the basis of fuzzy equality?

- **Residuated lattice** $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$
[V. Novák. On fuzzy type theory. *Fuzzy Sets and Systems*, 149:235-273, 2005.]

- **EQ-algebra**
[M. Dyba and V. Novák. EQ-logics: Non-commutative fuzzy logics based on fuzzy equality. *Fuzzy Sets and Systems*, 2011, sv. 172, 13–32.]

[Dyba, M., Novák, V., EQ-logics with delta connective. *Iranian Journal of Fuzzy Systems*, submitted.]

[V. Novák. EQ-algebra-based fuzzy type theory and its extensions. *Logic Journal of the IGPL*, 2011, 19, 512–542.]

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Definition

Non-commutative EQ-algebra is the algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

of type $(2, 2, 2, 0)$

- (E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a commutative idempotent monoid (i.e. \wedge -semilattice with top element $\mathbf{1}$) with the ordering: $a \leq b$ iff $a \wedge b = a$
- (E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and \otimes is isotone w.r.t. \leq

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Definition (continued)

(E3) $a \sim a = 1$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq c \sim (d \wedge b)$ (substitution)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (monotonicity)

(E7) $a \otimes b \leq a \sim b$ (boundedness)

Implication: $a \rightarrow b = (a \wedge b) \sim a$

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Special EQ-algebras

EQ-algebra is

- (a) good if $a \sim \mathbf{1} = a$
- (b) residuated if

$$(a \otimes b) \wedge c = a \otimes b \quad \text{iff} \quad a \wedge ((b \wedge c) \sim b) = a$$
- (c) involutive if $\neg\neg a = a$ (IEQ-algebra)
- (d) prelinear if for all $a, b \in E$ $\sup\{a \rightarrow b, b \rightarrow a\} = \mathbf{1}$.
- (e) lattice EQ-algebra if it is a lattice-ordered and for all $a, b, c, d \in E$

$$((a \vee b) \sim c) \otimes (d \sim a) \leq (d \vee b) \sim c$$
(ℓEQ-algebra)

Special EQ-algebras

A lattice EQ_Δ -algebra (ℓEQ_Δ -algebra)

$$\mathcal{E}_\Delta = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$$

- $\langle E, \wedge, \vee, \otimes, \sim, \mathbf{0}, \mathbf{1} \rangle$ is a good non-commutative and bounded ℓEQ -algebra.
- $\Delta \mathbf{1} = \mathbf{1}$
- $\Delta a \leq \Delta \Delta a$
- $\Delta(a \sim b) \leq \Delta a \sim \Delta b$
- $\Delta(a \wedge b) = \Delta a \wedge \Delta b$
- $\Delta a = \Delta a \otimes \Delta a$
- $\Delta(a \vee b) \leq \Delta a \vee \Delta b$
- $\Delta a \vee \neg \Delta a = \mathbf{1}$
- $\Delta(a \sim b) \leq (a \otimes c) \sim (b \otimes c)$
- $\Delta(a \sim b) \leq (c \otimes a) \sim (c \otimes b)$

Representation of ℓEQ_Δ -algebras

Lemma

If a good EQ-algebra \mathcal{E} satisfies

$$(a \rightarrow b) \vee (d \rightarrow (d \otimes (c \rightarrow ((b \rightarrow a) \otimes c)))) = \mathbf{1} \quad (1)$$

for all $a, b, c, d \in E$ then it is prelinear.

Theorem

Let \mathcal{E}_Δ be ℓEQ_Δ -algebra. The following are equivalent:

- (a) \mathcal{E}_Δ is subdirectly embeddable into a product of linearly ordered good ℓEQ_Δ -algebras.
- (b) \mathcal{E}_Δ satisfies (1).

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Why EQ-logics

EQ-logics — special class of many-valued logics
truth values form an EQ-algebra

- Equivalence as the basic connective instead of implication
- Proofs in equational style
- Even more general than MTL-logics

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Language

- Propositional variables p_1, p_2, \dots
- Connectives: \wedge (conjunction), $\&$ (fusion), \equiv (equivalence),
- Logical constant \top (true)

Implication:

$$A \Rightarrow B := (A \wedge B) \equiv A$$

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Logical axioms

$$(EQ1) \quad (A \equiv \top) \equiv A$$

$$(EQ2) \quad A \wedge B \equiv B \wedge A$$

$$(EQ3) \quad (A \circ B) \circ C \equiv A \circ (B \circ C), \quad \circ \in \{\wedge, \&\}$$

$$(EQ4) \quad A \wedge A \equiv A$$

$$(EQ5) \quad A \wedge \top \equiv A$$

$$(EQ6) \quad A \& \top \equiv A$$

$$(EQ7) \quad \top \& A \equiv A$$

$$(EQ8a) \quad ((A \wedge B) \& C) \Rightarrow (B \& C)$$

$$(EQ8b) \quad (C \& (A \wedge B)) \Rightarrow (C \& B)$$

$$(EQ9) \quad ((A \wedge B) \equiv C) \& (D \equiv A) \Rightarrow (C \equiv (D \wedge B)) \text{ (substitution)}$$

$$(EQ10) \quad (A \equiv B) \& (C \equiv D) \Rightarrow (A \equiv C) \equiv (B \equiv D) \text{ (congruence)}$$

$$(EQ11) \quad (A \Rightarrow (B \wedge C)) \Rightarrow (A \Rightarrow B) \text{ (monotonicity)}$$

Inference rules

Equanimity rule

*From A and $A \equiv B$ **infer** B*

Leibniz rule

*From $A \equiv B$ **infer** $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$*

$C[\mathbf{p} := A]$ denotes a formula resulting from C by replacing all occurrences of a variable \mathbf{p} in C by the formula A .

Semantics

Truth values

The set of truth values is a good non-commutative EQ-algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

Theorem (Completeness)

For every formula $A \in F_J$ the following is equivalent:

- (a)** $\vdash A$
- (b)** $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every good non-commutative EQ-algebra \mathcal{E} .

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Other EQ-logics

- Involutive EQ-logic (with double negation)
- Prelinear EQ-logic (stronger variant of the completeness theorem)
- EQ(MTL)-logic (equivalent with MTL-logic)

Not strong enough for development of the predicate EQ-logic!

- Basic EQ_Δ -logic (weaker variant of the completeness theorem)
- Prelinear EQ_Δ -logic

Theorem (Deduction)

For each theory T and formulas $A, B, C \in F_J$:

$$T \cup \{A \equiv B\} \vdash C \quad \text{iff} \quad T \vdash \Delta(A \equiv B) \Rightarrow C$$

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Prelinear EQ $_{\Delta}$ -logic

Language

The language of basic EQ-logic extended by unary connective Δ , binary connective \vee and logical constant \perp .

Negation $\neg A := A \equiv \perp$

Axioms (EQ1)–(EQ11) and

$$((((A \wedge B) \vee C) \equiv D) \& (F \equiv C)) \& (E \equiv A) \Rightarrow (D \equiv (F \vee (B \wedge E)))$$

$$\text{(EQ12)} \quad (A \vee B) \vee C \equiv A \vee (B \vee C)$$

$$\text{(EQ13)} \quad A \vee (A \wedge B) \equiv A$$

$$\text{(EQ14)} \quad (A \wedge \perp) \equiv \perp$$

$$\text{(EQ15)} \quad (A \Rightarrow B) \vee (D \Rightarrow (D \& (C \Rightarrow ((B \Rightarrow A) \& C))))$$

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$$\text{(EQ15)} \quad (A \Rightarrow B) \vee (D \Rightarrow (D \& (C \Rightarrow ((B \Rightarrow A) \& C))))$$

Prelinear EQ $_{\Delta}$ -logic

Axioms (continued)

$$(EQ\Delta 1) \quad \Delta A \Rightarrow \Delta\Delta A$$

$$(EQ\Delta 2) \quad \Delta(A \equiv B) \Rightarrow (\Delta A \equiv \Delta B)$$

$$(EQ\Delta 3) \quad \Delta(A \wedge B) \equiv (\Delta A \wedge \Delta B)$$

$$(EQ\Delta 4) \quad \Delta A \equiv (\Delta A \& \Delta A)$$

$$(EQ\Delta 5) \quad \Delta(A \vee B) \Rightarrow (\Delta A \vee \Delta B)$$

$$(EQ\Delta 6) \quad \Delta A \vee \neg \Delta A$$

$$(EQ\Delta 7) \quad \Delta(A \equiv B) \Rightarrow ((A \& C) \equiv (B \& C))$$

$$(EQ\Delta 8) \quad \Delta(A \equiv B) \Rightarrow ((C \& A) \equiv (C \& B))$$

Prelinear EQ_Δ -logic

Inference rules

- Equanimity rule
- Leibniz rule
- Necessitation rule

*From A **infer** ΔA*

Semantics

An ℓEQ_Δ -algebras in which (1) is satisfied.

Prelinear EQ $_{\Delta}$ -logic

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From A infer ΔA

Semantics

An ℓ EQ $_{\Delta}$ -algebras in which (1) is satisfied.

Prelinear EQ_{Δ} -logic

Theorem (Completeness)

For every formula $A \in F_J$ and every theory T the following is equivalent:

- (a)** $T \vdash A$
- (b)** $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every linearly ordered, ℓEQ_{Δ} -algebra \mathcal{E} .
- (c)** $e(A) = \mathbf{1}$ for every truth evaluation $e : F_J \longrightarrow E$ and every ℓEQ_{Δ} -algebra \mathcal{E} satisfying (1).

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Syntax

Language

- Object variables x, y, \dots .
- Set of object constants $\text{Const} = \{\mathbf{u}, \mathbf{v}, \dots\}$.
- Non-empty set of n -ary predicate symbols
 $\text{Pred} = \{P, Q, \dots\}$.
- Binary connectives $\wedge, \vee, \&, \equiv$ and unary connective Δ .
- Logical (truth) constants \top (true) and \perp (false).
- Quantifiers \forall, \exists .
- Auxiliary symbols: brackets.

Syntax

Terms

Object variables and object constants are terms.

Formulas

- If P is an n -ary predicate symbol and t_1, \dots, t_n are terms then $P(t_1, \dots, t_n)$ is atomic formula.
- Logical constants \top and \perp are (atomic) formulas.
- If A, B are formulas then $A \wedge B, A \vee B, A \& B, A \equiv B, \Delta A$ are formulas.
- If A is formula and x is an object variable then $(\forall x)A, (\exists x)A$ are formulas.

Semantics

Structure for language \mathcal{J}

$$\mathcal{M}^{\mathcal{E}} = \langle M, \mathcal{E}, \{r_P\}_{P \in \text{Pred}}, \{m_u\}_{u \in \text{Const}} \rangle$$

$\mathcal{E} = \langle E, \wedge, \vee, \otimes, \sim, \Delta, \mathbf{0}, \mathbf{1} \rangle$ is a non-commutative linearly ordered ℓEQ_{Δ} -algebra,

$r_P : M^n \longrightarrow E$ is n -ary relation,

$m_u \in M$.

Interpretation of terms and formulas

v — assignment of elements from M to variables

$$\mathcal{M}_v^{\mathcal{E}}(x) = v(x), \mathcal{M}_v^{\mathcal{E}}(\mathbf{u}) = m_u,$$

$$\mathcal{M}_v^{\mathcal{E}}(P(t_1, \dots, t_n)) = r_P(\mathcal{M}_v^{\mathcal{E}}(t_1), \dots, \mathcal{M}_v^{\mathcal{E}}(t_n)),$$

$$\mathcal{M}_v^{\mathcal{E}}(A \wedge B) = \mathcal{M}_v^{\mathcal{E}}(A) \wedge \mathcal{M}_v^{\mathcal{E}}(B),$$

$$\mathcal{M}_v^{\mathcal{E}}(A \vee B) = \mathcal{M}_v^{\mathcal{E}}(A) \vee \mathcal{M}_v^{\mathcal{E}}(B),$$

$$\mathcal{M}_v^{\mathcal{E}}(A \& B) = \mathcal{M}_v^{\mathcal{E}}(A) \otimes \mathcal{M}_v^{\mathcal{E}}(B),$$

$$\mathcal{M}_v^{\mathcal{E}}(A \equiv B) = \mathcal{M}_v^{\mathcal{E}}(A) \sim \mathcal{M}_v^{\mathcal{E}}(B),$$

$$\mathcal{M}_v^{\mathcal{E}}(\Delta A) = \Delta \mathcal{M}_v^{\mathcal{E}}(A), \mathcal{M}_v^{\mathcal{E}}(\top) = \mathbf{1}, \mathcal{M}_v^{\mathcal{E}}(\perp) = \mathbf{0},$$

$$\mathcal{M}_v^{\mathcal{E}}((\forall x)A) = \inf\{\mathcal{M}_{v'}^{\mathcal{E}}(A) \mid v' = v \setminus x\},$$

$$\mathcal{M}_v^{\mathcal{E}}((\exists x)A) = \sup\{\mathcal{M}_{v'}^{\mathcal{E}}(A) \mid v' = v \setminus x\}$$

Logical Axioms

(EQ1)–(EQ15), (EQ Δ 1)–(EQ Δ 8) plus

(EQ \forall 1) $(\forall x)A(x) \Rightarrow A(t)$ (t substituable for x in $A(x)$),

(EQ \exists 1) $A(t) \Rightarrow (\exists x)A(x)$ (t substituable for x in $A(x)$),

(EQ $\forall\Delta$ 2) $\Delta(\forall x)(A \Rightarrow B) \Rightarrow (A \Rightarrow (\forall x)B)$ (x not free in A),

(EQ \exists 2) $(\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow B)$ (x not free in B),

(EQ \forall 3) $(\forall x)(A \vee B) \Rightarrow ((\forall x)A \vee B)$ (x not free in B),

Inference Rules

- Equanimity rule
- Leibniz rule

*From $A \equiv B$ **infer** $C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]$,*

provided that the subformula \mathbf{p} is not in the scope of a quantifier in C .

- Necessitation rule
- Rule of Generalization

*From A **infer** $(\forall x)A$*

Model

Definition

Structure $\mathcal{M}^{\mathcal{E}}$ is a model of a theory T if $\mathcal{M}_v^{\mathcal{E}}(A) = \mathbf{1}$ holds for all axioms A of T .

Main properties

Lemma

- (i) $\vdash (\forall x)(A \Rightarrow B) \equiv (A \Rightarrow (\forall x)B)$, x not free in A
- (ii) $\vdash (\forall x)(B \Rightarrow A) \equiv ((\exists x)B \Rightarrow A)$, x not free in A
- (iii) $\vdash \Delta(\forall x)(A \Rightarrow B) \Rightarrow ((\forall x)A \Rightarrow (\forall x)B)$
- (iv) $\vdash (\forall x)(A \Rightarrow B) \Rightarrow ((\exists x)A \Rightarrow (\exists x)B)$

Theorem (Deduction)

For each theory T , closed formulas A, B and arbitrary formula C :

$$T \cup \{A \equiv B\} \vdash C \quad \text{iff} \quad T \vdash \Delta(A \equiv B) \Rightarrow C.$$

Completeness

Definition

- (i) Theory T is consistent if there is a formula A unprovable in T .
- (ii) T is linear (complete) if for every two formulas A, B ,
 $T \vdash A \Rightarrow B$ or $T \vdash B \Rightarrow A$.
- (iii) T is extensionally complete if for every closed formula $(\forall x)(A(x) \equiv B(x))$, $T \not\vdash (\forall x)(A(x) \equiv B(x))$ there is a constant \mathbf{u} such that $T \not\vdash (A_x[\mathbf{u}] \equiv B_x[\mathbf{u}])$

Completeness

Theorem

Every consistent theory T can be extended to a maximally consistent linear theory.

Theorem

Every consistent theory T can be extended to an extensionally complete consistent theory T .

Completeness

Theorem (Completeness)

- (a) *A theory T of the predicate first-order EQ-logic is consistent iff it has a safe model \mathcal{M} .*
- (b) $T \vdash A$ iff $T \models A$

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Conclusion

- Formal system of predicate EQ-logic with equivalence as main connective was developed.
- Δ -connective (deduction theorem) is indispensable for development of the first-order EQ-logic.

Current and future work:

- To extend the language of the predicate EQ-logic by the functional symbols and symbol \doteq for fuzzy equality between objects.
- To prepare a software (based on differential evolution) that finds examples of EQ-algebras with given properties.

Thank you for your attention.